Australian Government
Department of Defence
Defence Science and
Technology Organisation

Cramér-Rao bounds for target tracking

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Talk Outline

- What is CRB and why do we need it?
- CRB for nonlinear filtering
- CRB for jump Markov processes
- CRB for uncertain data association
- Multi-target CRB
- Sensor allocation using CRB
- Summary

What is Cramér-Rao bound?

- CR inequality provides a lower bound on the achievable mean-square estimation error.
- The CRB for *unbiased* estimators mainly in use (although the formulation for biased estimators is also available);
- We distinguish two cases:
  - ◇ *deterministic* parameter estimation
  - ◇ *stochastic* parameter estimation (a.k.a. *posterior* CRB)
- Existence of the CR bound not guaranteed.

Some history

- The CR inequality was first stated by Ronald Fisher (1925).
- Proven by Daniel Dugué (1937).
- Harold Cramér, C. R. Rao (independently) merely re-derived the bound (1945)!
- H. Van Trees (1968) introduced the bound to a wider engineering community.

Applications of the CR bound (tracking context)

- Theoretically possible to predict the best achievable 2nd-order error performance for a target tracking problem (before you develop an algorithm);
- Aid in a tracker design: one can assess the effects of approximations embedded in tracking algorithms (by comparing RMS errors with the bound);
- Sensor management applications:
  - radar scheduling;
  - spatial deployment of sonobuoys;
  - observer trajectories (bearings-only tracking, cooperative UAVs, etc)

Definition (static case)

- Suppose $\mathbf{x}$ is an unknown random parameter vector (dim n_x)
- $\mathbf{Z} = (\mathbf{z}_1, \ldots, \mathbf{z}_k)$ is a vector of measurement data
- Let $\hat{\mathbf{x}} = \mathbf{g}(\mathbf{Z})$ be an unbiased estimate of $\mathbf{x}$.
- The Cramér-Rao inequality:

$$\boxed{\mathbf{C} \triangleq \mathbb{E}\left\{[\mathbf{g}(\mathbf{Z}) - \mathbf{x}]\,[\mathbf{g}(\mathbf{Z}) - \mathbf{x}]^T\right\} \geq \mathbf{J}^{-1}}$$

- $\mathbf{J}$ is the (Fisher) information matrix with elements:

$$\mathbf{J}_{ij} = -\mathbb{E}\left[\frac{\partial^2 \ln p(\mathbf{x}, \mathbf{Z})}{\partial \mathbf{x}_i\, \partial \mathbf{x}_j}\right] \qquad (i, j = 1, \ldots, n_x)$$

Some properties of the bound

- Inequality $\mathbf{C} \geq \mathbf{J}^{-1}$ means that the difference $\mathbf{C} - \mathbf{J}^{-1}$ is a positive semi-definite matrix;

- Since $p(\mathbf{x}, \mathbf{Z}) = p(\mathbf{Z}|\mathbf{x}) \cdot p(\mathbf{x})$, the information matrix decomposed as:

$$\mathbf{J} = \mathbf{J}_z + \mathbf{J}_p$$

  where $\mathbf{J}_z$ represents the information obtained from the data and $\mathbf{J}_p$ represents the prior information

- If prior pdf $p(\mathbf{x})$ is a multivariate Gaussian with covariance $\mathbf{P}_0$, then $\mathbf{J}_p = \mathbf{P}_0^{-1}$

- The diagonal elements of $\mathbf{J}^{-1}$ are lower bounds of the corresponding mean-square error.

Nonlinear Filtering Problem (dynamic systems)

Notation:

- k is the discrete-time index
- $\mathbf{x}_k$ is target state vector at time k
- $\mathbf{z}_k^\ell$ is the measurement vector at time k from sensor $\ell = 1, \ldots, L$
- $\mathbf{w}_k$, $\mathbf{v}_k$ are independent white processes
- $\mathbf{f}_k(\cdot)$, $\mathbf{h}_k(\cdot)$ are nonlinear functions

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}) + \mathbf{w}_{k-1}$$

$$\mathbf{z}_k^\ell = \mathbf{h}_k^\ell(\mathbf{x}_k) + \mathbf{v}_k^\ell$$

for $k = 1, 2, 3, \ldots$

The assumption is that the initial state $\mathbf{x}_0$ has a known pdf $p(\mathbf{x}_0)$.

The CR bound for the Nonlinear Filtering Problem

- Research topic for about three decades:
  $\Rightarrow$ an excellent review by T. H. Kerr (1989)

- Tichavský et al. (1998): Riccati-like recursion for the calculation of $\mathbf{J}_k$.

$$\mathbf{J}_{k+1} = \mathbf{J}_p(k+1) + \sum_{\ell} \mathbf{J}_z^{\ell}(k+1)$$

  - ◇ $\mathbf{J}_p(k+1)$ is prior (or predicted) information matrix
  - ◇ $\mathbf{J}_z^{\ell}(k+1)$ is information matrix due to measurement from sensor $\ell = 1, \ldots, L$ at time k. Further on we assume $\ell = 1$ for simplicity.

The CR bound for the Nonlinear Filtering Problem (Cont'd)

- If process noise $\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_k)$, and $\boldsymbol{\Sigma}_k$ non-singular, then

$$\mathbf{J}_p(k+1) = \boldsymbol{\Sigma}_k^{-1} - \boldsymbol{\Sigma}_k^{-1}\mathbb{E}\{\mathbf{F}_k\}\left(\mathbf{J}_k + \mathbb{E}\{\mathbf{F}_k^T\boldsymbol{\Sigma}_k^{-1}\mathbf{F}_k\}\right)^{-1}\mathbb{E}\{\mathbf{F}_k^T\}\boldsymbol{\Sigma}_k^{-1}$$

where $\mathbf{F}_k = \left[\nabla_{\mathbf{x}_k}\left[\mathbf{f}_k(\mathbf{x}_k)\right]^T\right]^T$ is the Jacobian of $\mathbf{f}_k(\cdot)$.

- If measurement noise $\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$, and $\mathbf{R}_k$ non-singular, then

$$\mathbf{J}_z(k) = \mathbb{E}\left\{\mathbf{H}_k^T\mathbf{R}_k^{-1}\mathbf{H}_k\right\}$$

where $\mathbf{H}_k = \left[\nabla_{\mathbf{x}_k}\left[\mathbf{h}_k(\mathbf{x}_k)\right]^T\right]^T$ is the Jacobian of $\mathbf{h}_k(\cdot)$.

Nonlinear Filtering: Deterministic case

- In the absence of process noise, i.e. $\mathbf{w}_k = 0$, target state $\mathbf{x}_k$ is an unknown deterministic parameter (knowing $\mathbf{x}_0$ we can compute $\mathbf{x}_k$ for any k);

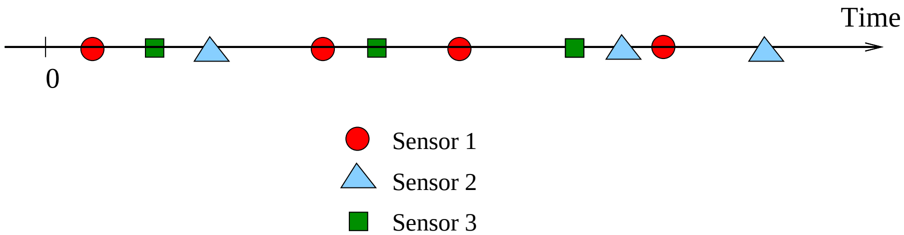
- The expectation operator $\mathbb{E}$ disappears; a simple recursive formula [Taylor, 1979]:

$$\boxed{\mathbf{J}_{k+1} = \left(\mathbf{F}_k^{-1}\right)^T \mathbf{J}_k \mathbf{F}_k^{-1} + \mathbf{H}_{k+1}^T \mathbf{R}_{k+1}^{-1} \mathbf{H}_{k+1}}$$

- *Observation:* This is identical to the covariance matrix propagation formula for the Extended Kalman filter! There is only one difference: here we use *true* values of $\mathbf{x}_k$ to evaluate Jacobians $\mathbf{F}_k$ and $\mathbf{H}_k$.

Examples: Bearings-only tracking

- Bearings measurements collected asynchronously by distributed sensors



- Target moving with a (nearly) constant velocity (linear dynamics);

$$\mathbf{x}_k = \begin{bmatrix} x_k & \dot{x}_k & y_k & \dot{y}_k \end{bmatrix}^T$$

- Sensors are mobile; sensor state vector is known:

$$\mathbf{x}_k^\ell = \begin{bmatrix} x_k^\ell & \dot{x}_k^\ell & y_k^\ell & \dot{y}_k^\ell \end{bmatrix}^T, \quad \ell \in \{1, 2, \ldots, L\}$$

Examples: Bearings-only tracking (Cont'd)

- Measurement equation (nonlinear)

$$\boxed{z_k^{\ell_k} = h_k^{\ell_k}(\mathbf{x}_k) + v_k^{\ell_k}}, \qquad h_k^{\ell_k}(\mathbf{x}_k) = \text{arctan}\,\frac{y_k - y_k^{\ell_k}}{x_k - x_k^{\ell_k}}$$

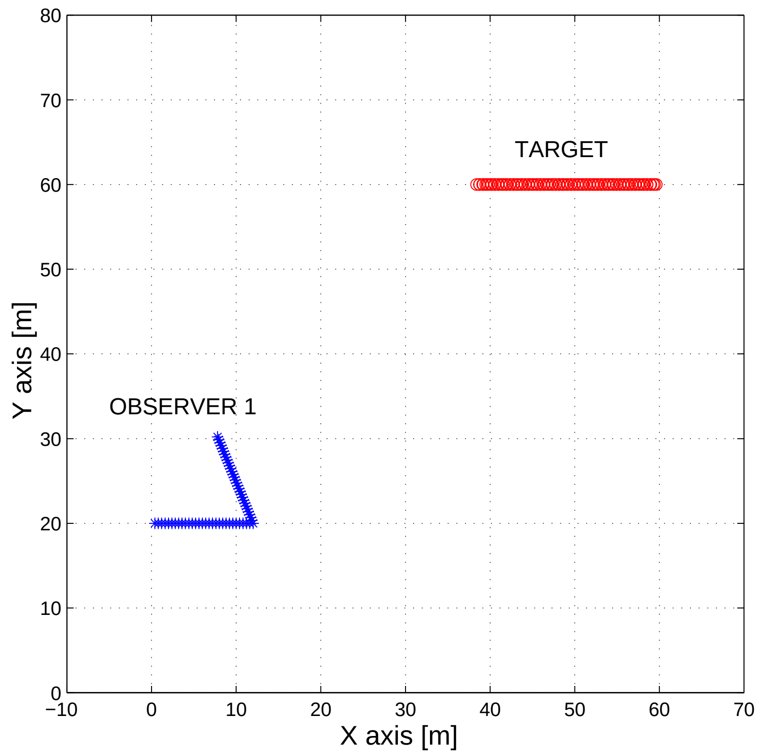
- $z_k^{\ell_k}$ is a measurement from sensor ℓ_k at time t_k;

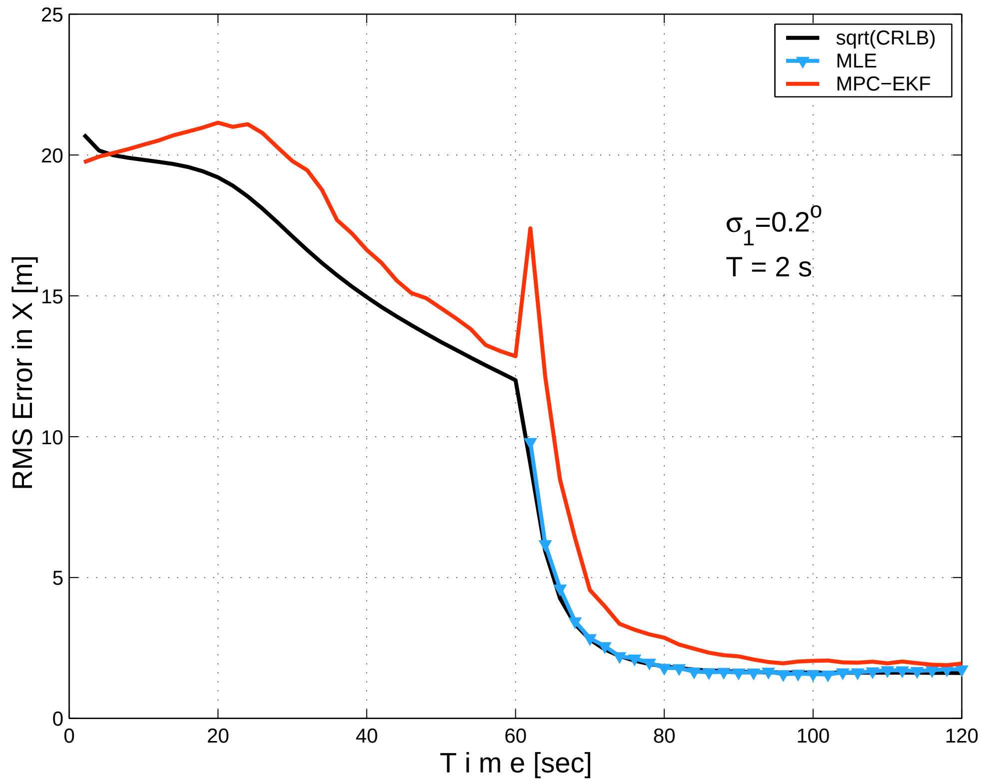
- $v_k^{\ell_k}$ is measurement noise in sensor ℓ_k: zero-mean white Gaussian, with variance $R^{\ell_k} = \sigma_{\ell_k}^2$.

- Estimation problem:
  Given sensor messages $\mathcal{M}_k = \{(t_i, \mathbf{x}_i^{\ell_i}, z_i^{\ell_i})\}$ ($i = 1, \ldots, k$), estimate $\mathbf{x}_k$.

Examples: Bearings-only tracking (Cont'd)

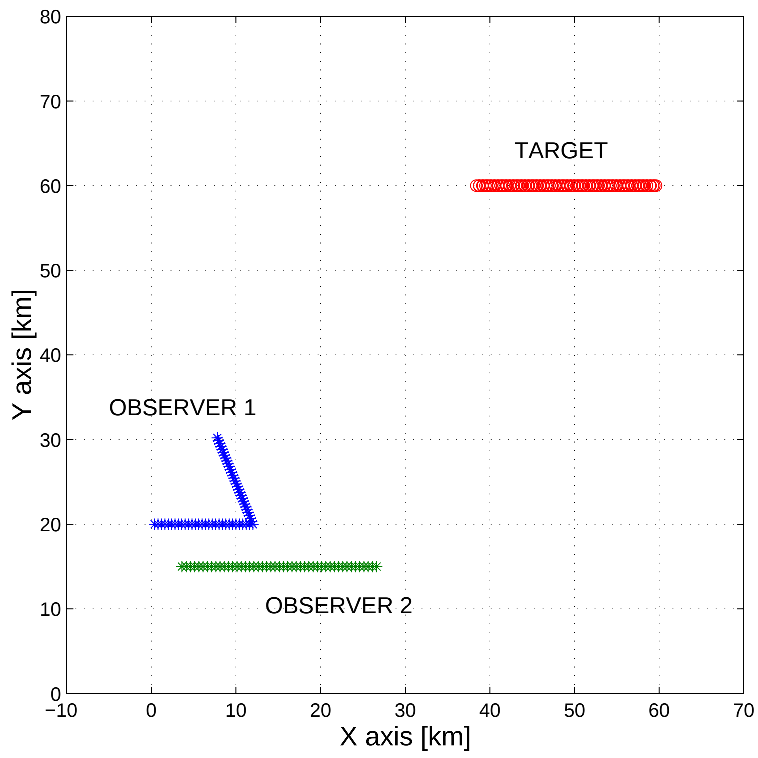
Single mobile sensor (must manoeuvre to observe the target state)

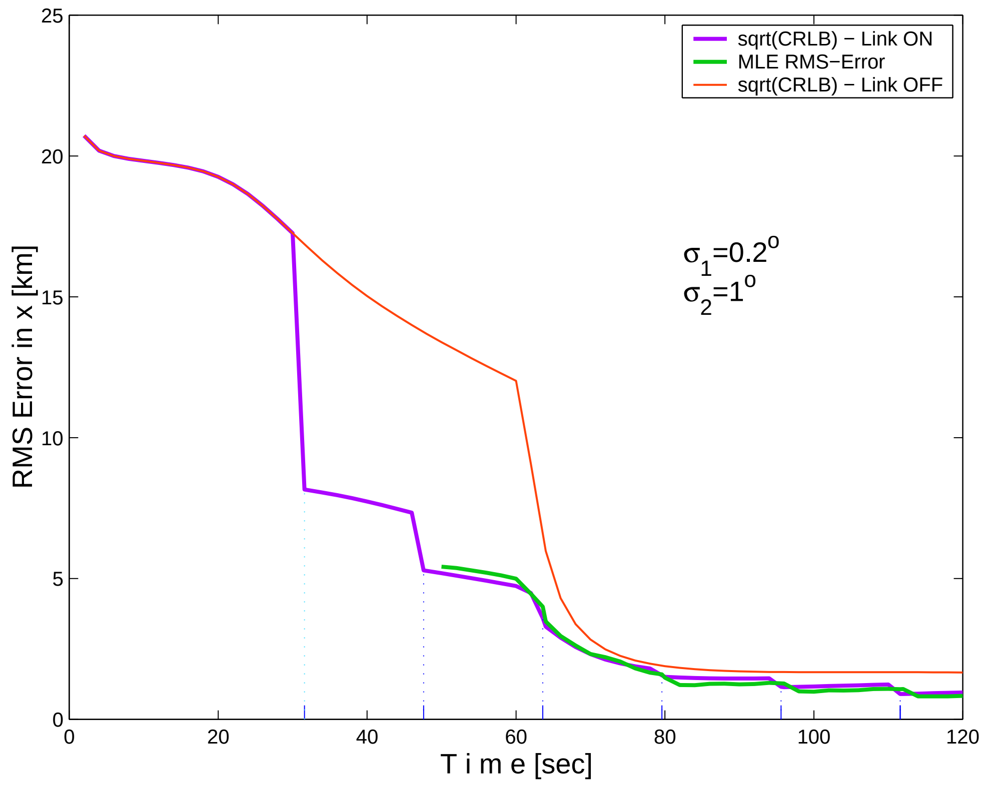




Examples: Bearings-only tracking (Cont'd)

Two Mobile sensors: Sensor 1 as before; Sensor 2 reports only at: 31.6s, 47.6s, 63.6s, 79.6s, 95.6s, 111.6s





Examples: Tracking a Ballistic Object on Re-entry

- Problem:

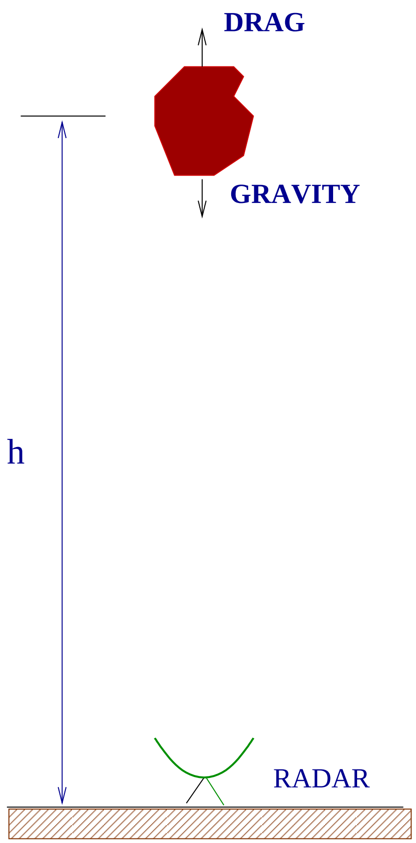
  Sequential estimation of kinematic parameters (position, velocity) of a ballistic object re-entering the atmosphere

- Practical applications: Surveillance for missile defence (e.g. scud missiles)

- Problem *difficult* due to the nonlinear object dynamics;

- Long history [Athans et. al. 1968; Mehra 1971; Gelb 1974; Austin 1981; Zarchan 1994; Julier et al. 2000]

Ballistic Object on Re-entry: Dynamics



- 1D (vertical) motion
- Only two forces act upon the object: drag (air resistance) and gravity
- Differential equations:

$$
\begin{aligned}
\dot{h} &= -v \\
\dot{v} &= \frac{\rho(h) \cdot g \cdot v^2}{2\beta} - g
\end{aligned}
$$

where

- h - object height;
- v - object velocity;
- β - ballistic coefficient (depends on mass, shape, cross-sec.);

$\rho(h) = \gamma \cdot e^{-\eta h}$ (air density);

$g = 9.81\text{m/s}^2$

Ballistic Object on Re-entry: Dynamics & measurements

- State vector $\mathbf{x}_k = [h_k \;\; v_k \;\; \beta_k]^T$;

- Using Euler approx. with a small integration step τ

$$\boxed{\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k) + \mathbf{w}_k}$$

  where $\mathbf{f}_k(\mathbf{x}_k)$ is nonlinear due to drag $D(\mathbf{x}_k) = \frac{g \cdot \rho(\mathbf{x}_k[1]) \cdot \mathbf{x}_k^2[2]}{2\mathbf{x}_k[3]}$

- Process noise: $\mathbf{w}_k \sim \mathcal{N}(0, \mathbf{\Sigma})$

- Radar is measuring target height (range) every $T \geq \tau$ seconds;

- Measurement equation is linear:

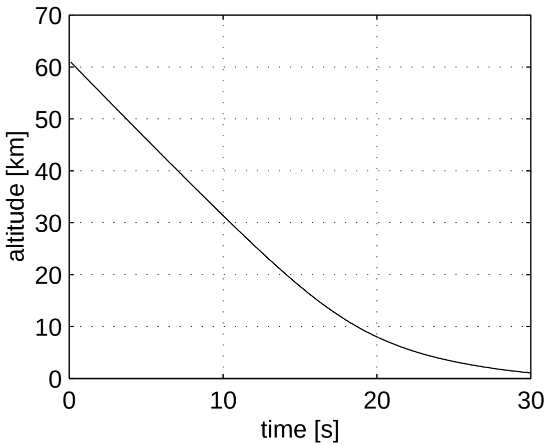
$$\boxed{z_k = \mathbf{H}\mathbf{x}_k + v_k}$$

  where $\mathbf{H} = [1 \;\; 0 \;\; 0]$ and $v_k \sim \mathcal{N}(0, R = \sigma_r^2)$.

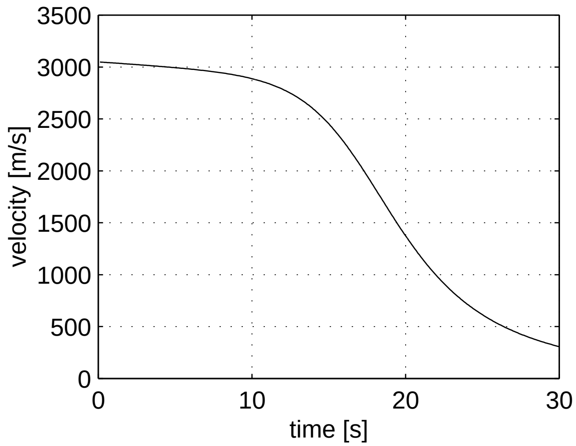
Ref: B. Ristic, S. Arulampalam, N. Gordon, *Beyond the Kalman filter*, 2004 (chapter 5).

Ballistic Object on Re-entry: Trajectory

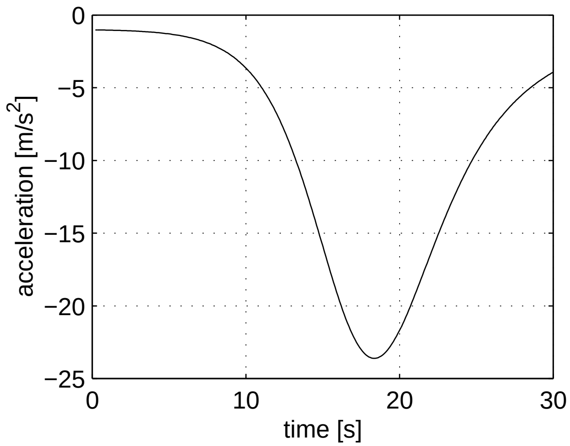
- $h_0 = 60960$ m;
- $v_0 = 3048$ m/s;
- $\beta_0 = 23948$ kg/ms^2 (corresp. mass of 500 kg)



Altitude



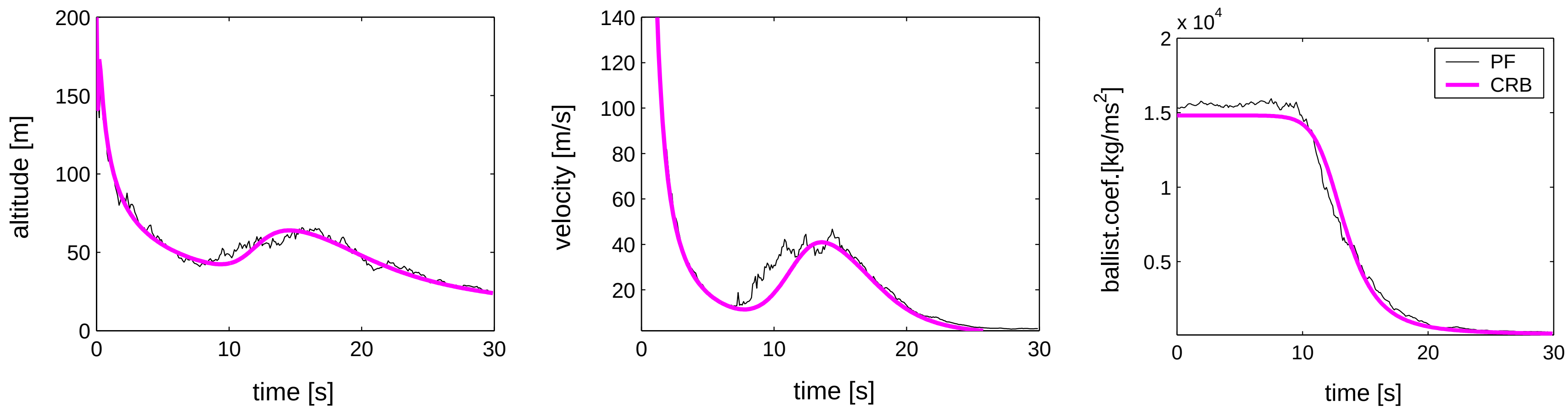
Velocity



Acceleration

Ballistic Object on Re-entry: CR bound

- $R = (200\text{m})^2$;
- $\sigma_\beta = 7184$ kg/ms^2

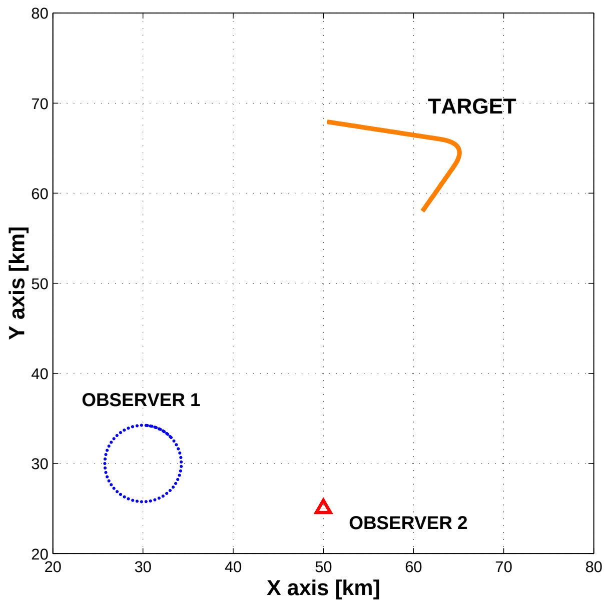


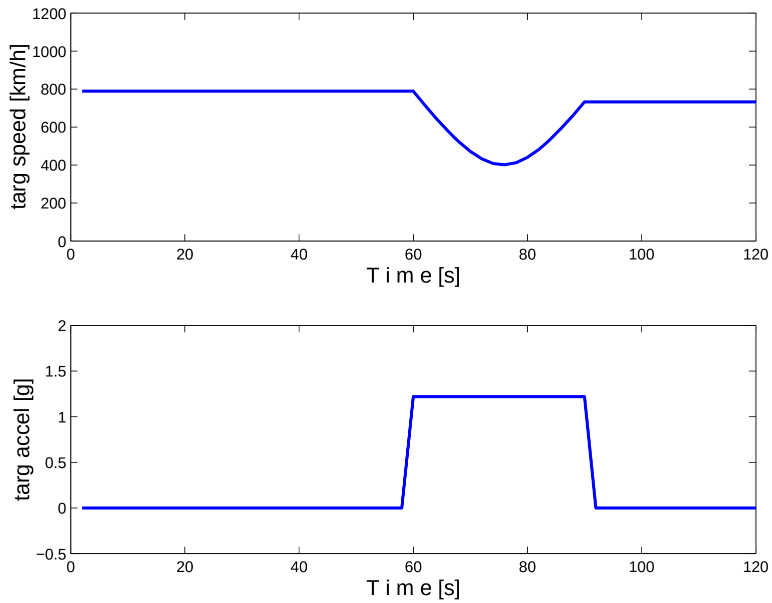
Altitude Velocity Ballistic coeff.

CRB for switching dynamic models

- Object motion sometimes must be modelled using more than a single dynamic model;

- Typical motion models: constant velocity, constant acceleration, coordinated turn, etc.





Switching Dynamic model

- Multiple switching linear dynamic models with additive Gaussian noise:

$$\mathbf{x}_{k+1} = \mathbf{F}_k(r_{k+1})\mathbf{x}_k + \mathbf{w}_k(r_{k+1})$$

- r_{k+1} specifies the target motion model (or regime) which is in effect during the time interval $(t_k, t_{k+1}]$;

- $\mathbf{w}_k(r_{k+1}) \sim \mathcal{N}(0, \Sigma_k(r_{k+1}))$;

- The evolution of motion model sequence is modelled by a time-homogeneous Markov chain with known:
  - ◇ transitional probabilities

$$\pi_{ij} \triangleq \mathbb{P}\{r_{k+1} = j | r_k = i\}, \qquad i, j \in S \triangleq \{1, 2, \ldots, s\}$$

  - ◇ initial motion model probabilities:

$$p_1(i) \triangleq \mathbb{P}\{r_1 = i\}, \qquad i \in S$$

- Required to estimate both $\mathbf{x}_k$ (continuous-valued) and r_k (discrete-valued): **Hybrid estimation!**

Error Bounds for switching dynamic models

- Impossible to derive exact Cramer-Rao lower bounds

  Requires differentiation of terms such as $\log p(r_{k+1}|r_k)$

- **Alternatives:**

  1. Explore more general bounds than the Cramer-Rao bound

     e.g. Bhattacharya, Bobovsky-Zakai, Weiss-Weinstein lower bounds

     Problem: computationally expensive!

  2. Develop an approximate Cramer-Rao lower bound

     a. Conditioning on the regime sequence (i.e. enumeration bound)

     b. Using best fitting Gaussian distributions [Hernandez, Ristic, Farina, 2005]

Conditioning on the regime sequence (enumeration bound)

- Let: $\rho_k^n \triangleq (r_1^n, \ldots, r_k^n)$ be n-th regime sequence ($n = 1, 2, \ldots, s^k$)
- Then easily shown: $$\mathbb{E}\left\{[\hat{\mathbf{x}}_k - \mathbf{x}_k]\,[\hat{\mathbf{x}}_k - \mathbf{x}_k]^T\right\} \geq \sum_{n=1}^{s^k} \mathbb{P}(\rho_k^n) \cdot [J_k^n]^{-1}$$
- The RHS gives the *enumeration* bound
- $\mathbb{P}(\rho_k^n)$ is the (prior) probability of sequence ρ_k^n; can be computed knowing initial $p_1(i)$ and transitional π_{ij} regime probabilities.
- J_k^n is the (Fisher) information matrix conditional on sequence ρ_k^n:

$$\mathbf{J}_k^n = \underbrace{\left[\Sigma_{k-1}(r_k^n) + F_{k-1}(r_k^n)[J_{k-1}^n]^{-1}F_{k-1}(r_k^n)^T\right]^{-1}}_{\mathbf{J}_p^n(k)} + \mathbf{J}_z(k)$$

Conditioning on ρ_k^n: Optimistic bound

- Each $\left[\mathbf{J}_k^n\right]^{-1}$ gives the error covariance bound for a **known** manoeuvre sequence ρ_k^n;
$\Rightarrow$ the resulting CR bound is overly optimistic!

- Demonstration of this over-optimism with simple example:

  - ◇ S1: target in either CT or NCV model (manoeuvring)

  - ◇ S2: target always in NCV model

  - ◇ measurements linear in target state in both cases: hence $J_z(k)$ same

- We expect the CRB for S1 (manoeuvring target) to be higher as a consequence of additional uncertainty due to model switching.

Conditioning on ρ_k^n: Optimistic bound demonstration

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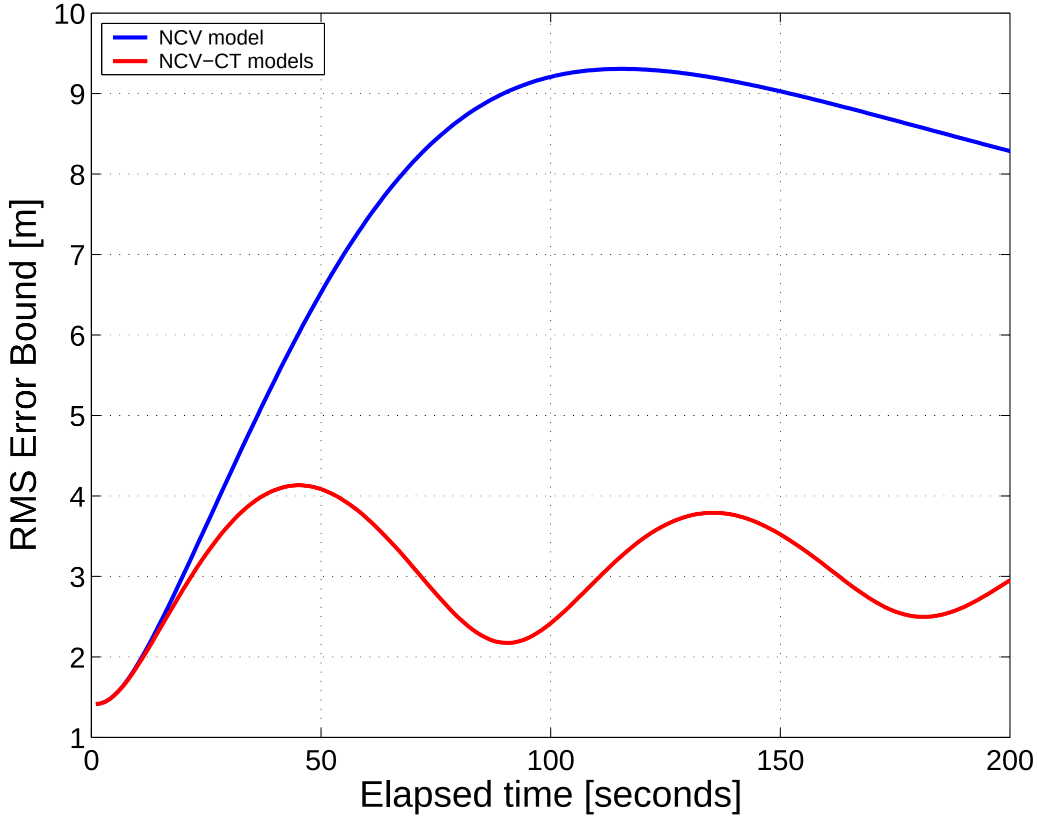


Switching Dynamic models: Best fitting Gaussian

- Original model (MODEL 1):

$$\mathbf{x}_{k+1} = \mathbf{F}_k(r_{k+1})\mathbf{x}_k + \mathbf{w}_k(r_{k+1}) \text{ with } \mathbf{w}_k(r_{k+1}) \sim \mathcal{N}(0, \mathbf{\Sigma}_k(r_{k+1}))$$

- Replace with a best-fitting Gaussian (BFG) approximation (MODEL 2):

$$\mathbf{x}_{k+1} \approx \mathbf{\Phi}_k\mathbf{x}_k + \epsilon_k \quad \text{with} \quad \epsilon_k \sim \mathcal{N}(0, \mathbf{Q}_k)$$

- $\mathbf{\Phi}_k$ and $\mathbf{Q}_k$ chosen so that:

$$\mathbb{E}[\mathbf{x}_k|\text{MODEL 1}] = \mathbb{E}[\mathbf{x}_k|\text{MODEL 2}] \quad \text{for all } k$$

$$Cov[\mathbf{x}_k|\text{MODEL 1}] = Cov[\mathbf{x}_k|\text{MODEL 2}] \quad \text{for all } k$$

- $\mathbf{Q}_k$ must also be positive definite (being a covariance)

Switching Dynamic models: Best fitting Gaussian (Cont'd)

- The BFG-CRB is then simply computed using the Riccati-like recursion:

$$\mathbf{J}_{k+1} = \left(\mathbf{Q}_k + \mathbf{\Phi}_k \mathbf{J}_k^{-1} \mathbf{\Phi}_k^T\right)^{-1} + \mathbf{J}_z(k+1)$$

- Initialisation:
  - ◇ Assuming that the prior pdf is: $\mathbf{x}_0 \sim N(\bar{\mathbf{x}}_0, \mathbf{P}_0)$, set:

$$\varepsilon_0 = \bar{\mathbf{x}}_0 \qquad \mathcal{C}_0 = \mathbf{P}_0$$

  - ◇ Determine mode probabilities:
    - ∗ define: $p_k(r) \triangleq \mathbb{P}(r_k = r)$, for $r = 1, \ldots, s$
    - ∗ determine: $p_k(r) = \sum_{j=1}^{s} \pi_{jr} p_{k-1}(j) \qquad$ for $k = 2, 3 \ldots$

BFG Distribution – General Recursion

- **STEP 1:** determine $\boldsymbol{\Phi}_k$ as follows:

$$\boldsymbol{\Phi}_k = \sum_{r=1}^{s} \mathbf{F}_k(r)\, p_{k+1}(r)$$

- **STEP 2:** determine $\mathcal{C}_{k+1}$ as follows:

$$\mathcal{C}_{k+1} = \sum_{r=1}^{s} p_{k+1}(r) \left[\mathbf{F}_k(r) \left(\mathcal{C}_k + \varepsilon_k \varepsilon_k^T \right) \mathbf{F}_k^T(r) + \boldsymbol{\Sigma}_k(r) \right] - \boldsymbol{\Phi}_k \varepsilon_k \varepsilon_k^T \boldsymbol{\Phi}_k^T$$

- **STEP 3:** determine $\mathbf{Q}_k$ as follows:

  (guaranteed $\mathbf{Q}_k \geq 0$)

$$\mathbf{Q}_k = \mathcal{C}_{k+1} - \boldsymbol{\Phi}_k \mathcal{C}_k \boldsymbol{\Phi}_k^T$$

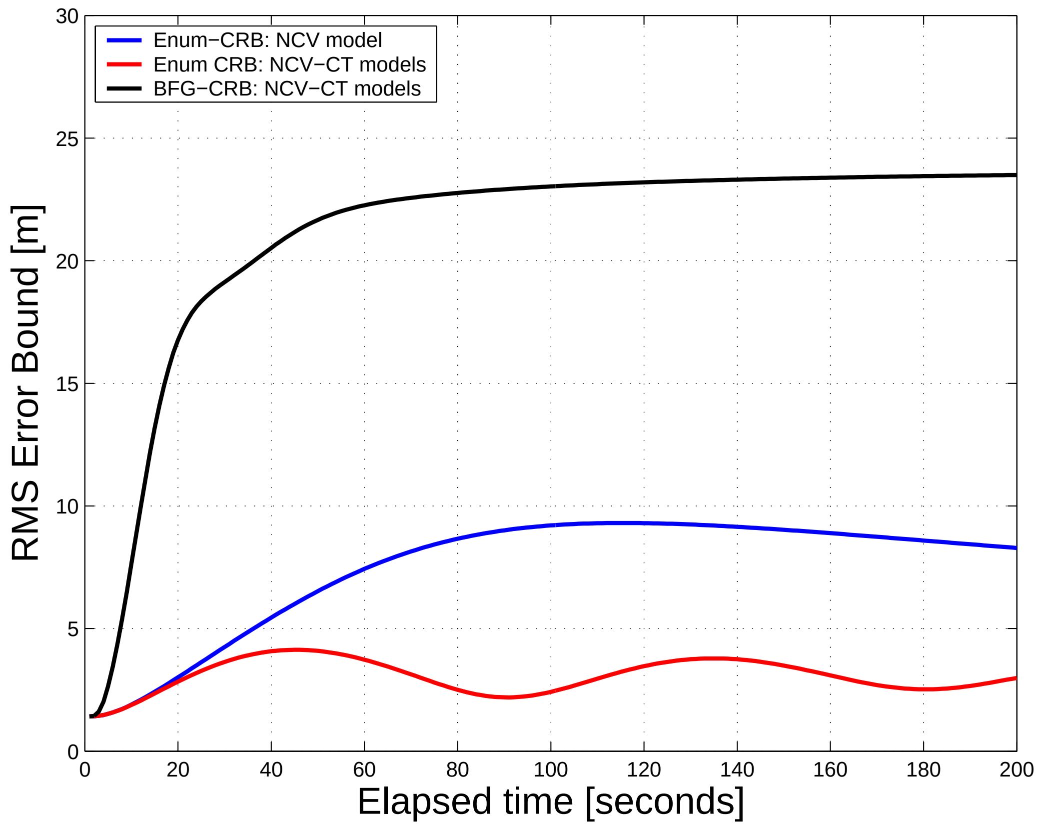
- **STEP 4:** determine ε_{k+1} as follows:

$$\varepsilon_{k+1} = \boldsymbol{\Phi}_k \varepsilon_k$$

- **STEP 5:** set: $k \longrightarrow (k+1)$ and repeat from **STEP 1**

BFG CR Bound demonstration

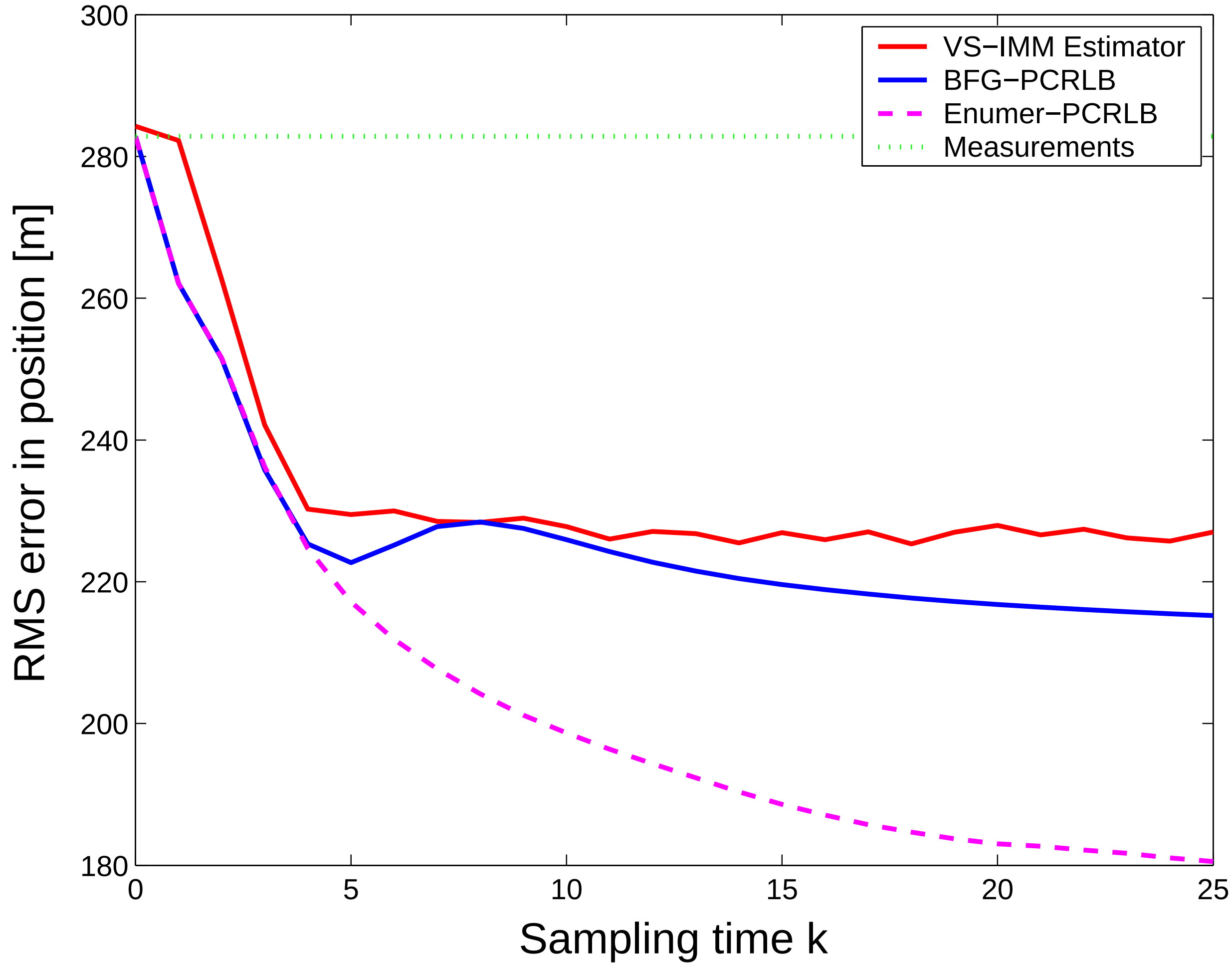
BFG approximation incorporates uncertainty due to model switching



Verification of the BFG approximation

- Aim: Compare the theoretical bound with empirical RMS error performance
- We simulate a target switching between CV or CA models (no process noise);
- Transition probabilities: $\pi_{ii} = 0.9$ for $i = 1, 2$
- Sampling time $T = 3$ seconds
- Measurements of Cartesian coordinates; error standard deviations: $\sigma_x = \sigma_y = 200$ m
- Comparison between:
  - Two theoretical CR bounds (BFG bound and Enumeration bound)
  - Empirical RMS error of an IMM filter; obtained via Monte Carlo simulations.

Verification of the BFG approximation (Cont'd)



The effect of $P_d < 1$ and $P_{fa} > 0$

- Most sensors characterised by $P_d < 1$ and $P_{fa} > 0$
  $\Rightarrow$ Uncertainty in measurement origin

- This type of uncertainty affects only $J_z(k)$ in: $\mathbf{J}_k = \mathbf{J}_p(k) + \mathbf{J}_z(k)$

- Several contributions since 1990 (more than 10 publications, Jauffret, Bar-Shalom, Zhang, Willet, Hernandez, Farina, Ristic, etc)

- The most comprehensive treatment (captures all previous developments) is the *measurement sequence conditioning* approach:

  Hernandez, Farina, Ristic, ”A PCRLB for tracking in cluttered environments: A measurement sequence conditioning approach”, to appear in IEEE Trans AES, 2006.

Measurement sequence conditioning

- Measurements sequence: $M_{1:k} = \{m_1, m_2, \dots, m_k\}$
- m_i is the number of measurements received at time $i = 1, \dots, k$. $m_i \in \{0, 1, 2, \dots\}$
- The CR inequality is then:

$$\mathbb{E}\left\{[\hat{\mathbf{x}} - \mathbf{x}]\,[\hat{\mathbf{x}} - \mathbf{x}]^T\right\} \geq \sum_{M_{1:k}} \mathbb{P}(M_{1:k})\,\mathbf{J}_k^{-1}(M_{1:k})$$

- $\mathbb{P}(M_{1:k})$ can be computed knowing:
  - the probability of detection P_d
  - the expected number of false measurements in the gate (Poisson model)

Measurement sequence conditioning (Cont'd)

- Information matrix as always have two components:

$$\mathbf{J}_k(M_{1:k}) = \mathbf{J}_p(k : M_{1:k-1}) + \mathbf{J}_z(k : m_k)$$

- Under some reasonable assumptions (rectangular gates, diagonal measurement matrix $\mathbf{R}_k$) we obtain:

$$\boxed{\mathbf{J}_z(k : m_k) = q_k(m_k)\ \mathbb{E}\{\mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k\}}$$

  where $q_k(m_k)$ is the information reduction factor (needs to be computed numerically);

- If $P_d = 1$ and $P_{fa} = 0$, then $m_k = 1$ and $q_k(1) = 1$ (see slide 10).

Measurement sequence conditioning: No false alarms

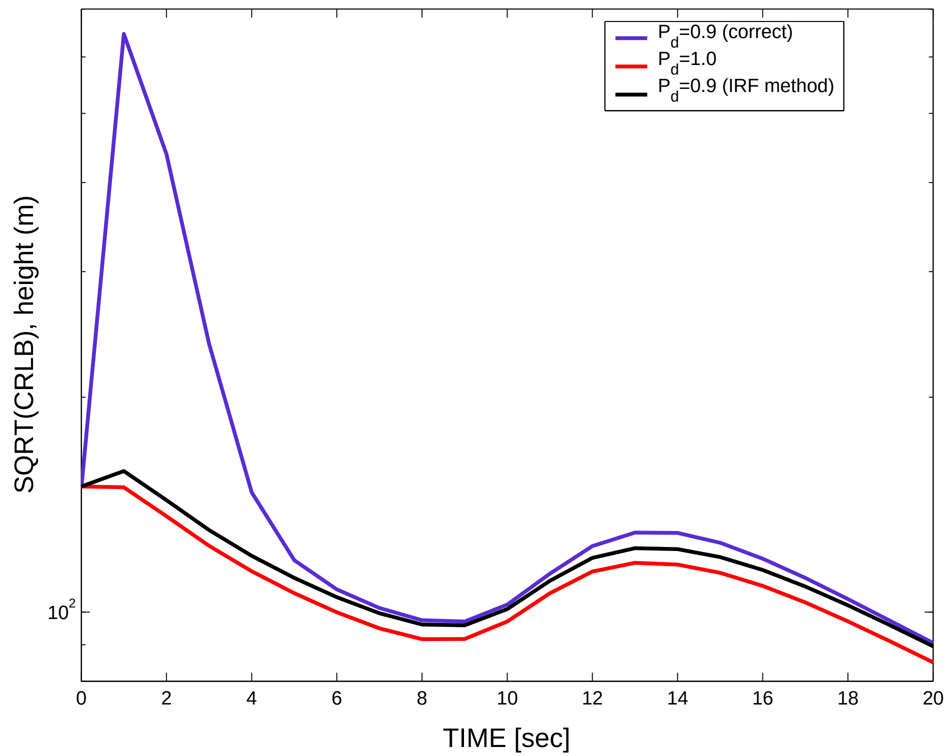
- $m_k \in \{0, 1\}$
- Sequence $M_{1:k}$ becomes a "detection/miss" sequence, so that:

$$\mathbf{J}_z(k : m_k) = \begin{cases} 0 & \text{if } m_k = 0, \\ \mathbb{E}\{\mathbf{H}_k^T \mathbf{R}_k^{-1} \mathbf{H}_k\} & \text{if } m_k = 1. \end{cases}$$

- The resulting bound first proposed in: Farina, Ristic, Timmoneri, "Cramér-Rao bound for nonlinear filtering with $P_d < 1$ and its application to target tracking", IEEE Trans SP, vol.50, 2002.
- When the false alarm rate is small (e.g. average number of false detections in the gate is below 0.1), the CR bound mainly influenced by $P_d < 1$.

The influence of $P_d < 1$

Tracking a ballistic object on re-entry (slide 20)



Ref: M. Hernandez, B. Ristic, A. Farina,L. Timmoneri, *IEEE Trans. SP*, vol.52, 2004

Multiple target tracking

- Notoriously difficult if multiple targets appear and disappear at random: the problem requires joint detection and tracking; Cramér-Rao bound not a suitable tool!

- If we assume that $L \geq 1$ targets exist in a surveillance region during the observation period, possible to formulate a CRB: Hue et al. [IEEE AES 2006], Tharmarasa et al [IEEE AES 2006].

- An analytic expression for multi-target CR bound in the framework of track-before-detect (ultimate bound)
  - ◇ Ref: B. Ristic, A. Farina, M. Hernandez, "Cramér-Rao lower bound for tracking multiple targets", *IEE Proc. Radar, Sonar, Navigation*, Vol.151, 2004.
  - ◇ Depends on SNR, sensor resolution, point-spread function and target kinematics.
  - ◇ Directly applicable to Wireless Sensor Networks (WSN)

Example: Wireless network of acoustic sensors

- State vector: $\mathbf{x}_{k,i} = [x_{k,i}\ \dot{x}_{k,i}\ y_{k,i}\ \dot{y}_{k,i}\ A_{k,i}]^T$;
  $i = 1, 2, \dots$ is target index

- Target motion nearly CV

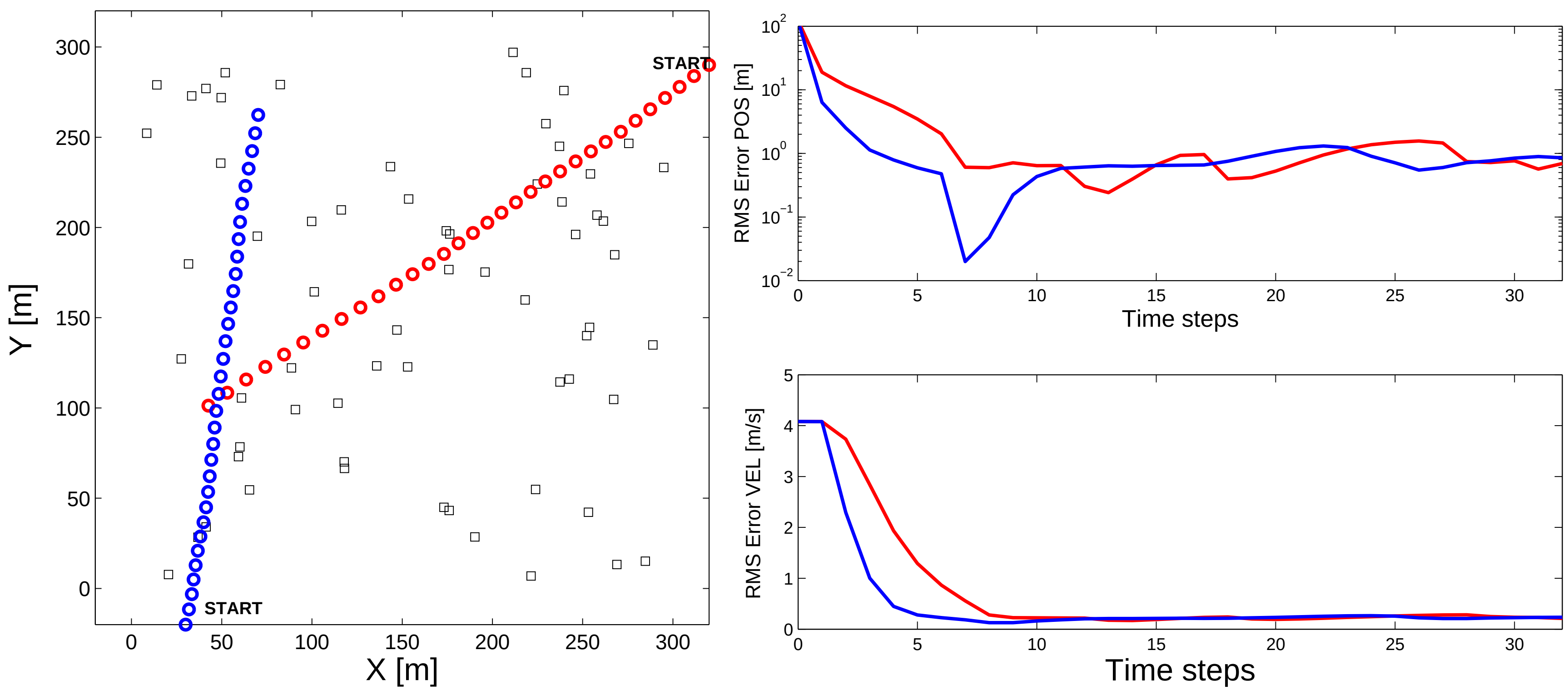
- Location of sensor j is: (X^j, Y^j), $j = 1, 2, \dots, N_s$

- Measurements of sound intensity (at sensor j):

$$z_k^j = \sum_i \frac{A_{k,i}}{\sqrt{(X^j - x_{k,i})^2 + (Y^j - y_{k,i})^2}} + v_k^j$$

Example: Wireless network of acoustic sensors (Cont'd)

- The (sound) intensity of the blue target is 3 dB higher
- Easy to include the effects of quantisation, and to predict the required sensor density.



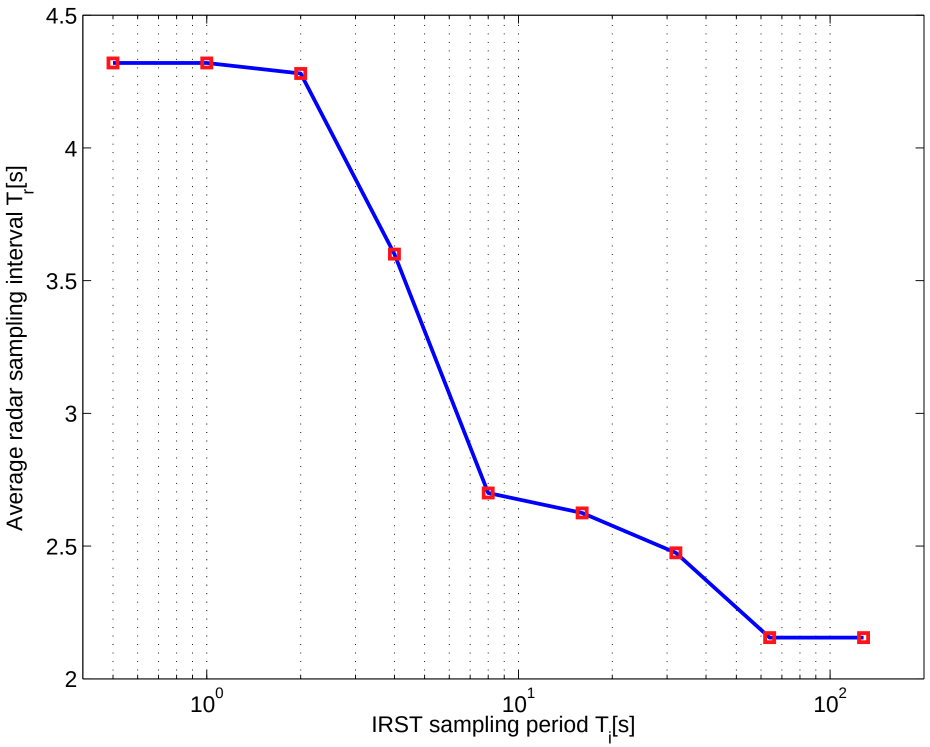
A Sensor Management Application

- Context:
  - ◇ Tracking of an anti-ship missile using a combination of a phased-array radar and an IRST sensor.
  - ◇ The IRST passively scans the horizon at a constant scanning interval in order to detect low altitude threats; each detection serves as an alert to allocate and cue the radar.

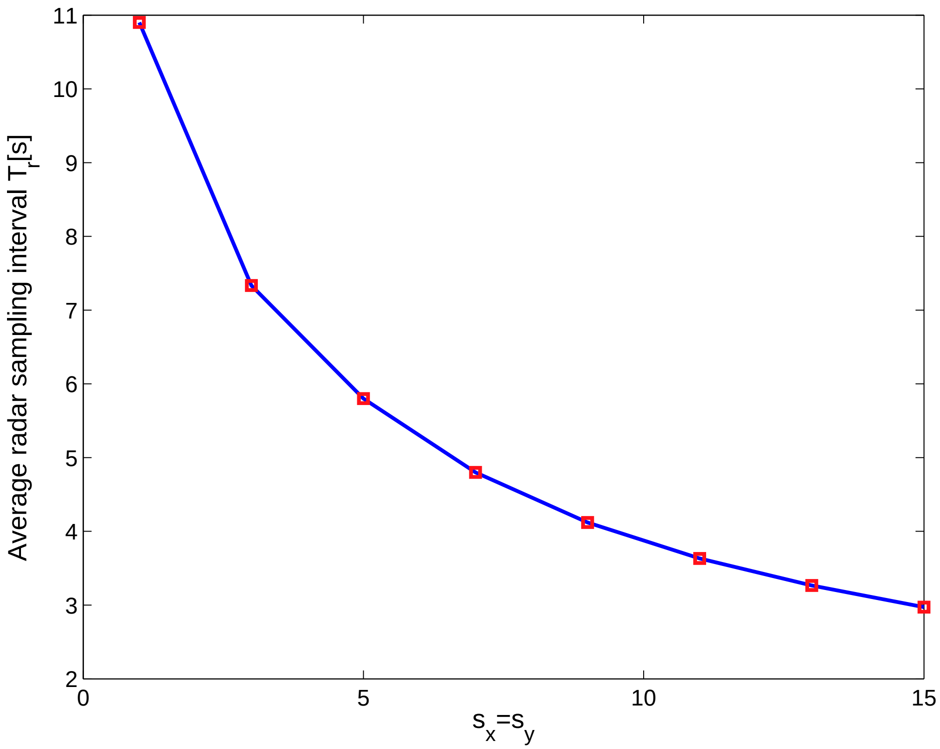
- The Cramér-Rao bound analysis applied to predict an average radar allocation requirements as a function of: target manoeuvrability, sensor accuracy, positional estimation accuracy.

Average radar update time

Versus (a) IRST sampling interval; (b) missile manoeuvrability



(a)



(b)

Summary

- Cramér-Rao bounds enable us to quantify the (best achievable) tracking error performance;
- A useful tool for tracker design, algorithm assessment, sensor management, etc.
- Significant progress made in the last few years on the CRB development for tracking
- Shortcomings:
  - impossible to compute in all situations (e.g. appearance of targets, switching models)
  - in some cases cannot be achieved by any practical estimator

Future work

- Multi-target tracking, hard constraints, comparison of *ultimate* bound with the *thresholding* bound, etc.

- Explore other variance bounds
  (Bhattacharya, Bobovsky-Zakai, Weiss-Weinstein, Barankin, etc)