Fixing Historical Misuse of Shaping Filters in Faulty Simulations of Maneuvering Missiles *

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Abstract

The existence of several fairly recent improper utilizations of shaping filters in a variety of different engineering applications motivated this paper as a cautionary warning. The warning consists of two distinct parts: (1) explaining what theoretical restrictions have been violated, and (2) providing explicit pointers to a few applications (such as in certain missile guidance simulations) where such dangerous violations have historically already occurred. We focus on this as a prelude to their remedy or at least to raise awareness to avoid any repeated violations. An alternate approach is discussed that rigorously supports comparable simulation objectives.

1 Introduction

The presence of several improperly invoked shaping filters in a smattering of varied engineering applications motivated the issuance of this cautionary warning so that others will be aware of and may thus avoid these traps that evidently have already ensnared so many. First, the motivation for using shaping filters, both historically and recently (also see [68, App.]), is briefly reviewed in Sec. 2. Certain philosophical questions that were previously asked and answered by pioneering theoreticians concerning the appropriateness of the shaping filter representation for specific power spectra sought are also reviewed in this section. These questions and their associated answers constitute the crux of what can be validly achieved using shaping filters.

New insights are offered in Sec. 2 into exactly where the lines of demarcation occur within the limits of applicability of fairly familiar input-output autocorrelation and spectral density relations for linear systems that in turn supports the shaping filter formalism being validly invoked. This elucidation of the limits of shaping filter applicability is the main constructive thrust of this paper. The obvious key requirement is established that, for valid use of shaping filters, the associated linear system must be strict sense stable and thus correspond to a stationary process. That a shaping filter has time invariant parameters and is realizable is not sufficient to satisfy this fundamental requirement. It will be demonstrated here that there have already been numerous unfortunate violations and misapplications of shaping filters along these lines that indicate an apparent lack of awareness of this previously unstated or underemphasized requirement. A brief overview summary of related supporting statistical notions and examples along these same lines is undertaken in Sec. 3. Several misapplications of shaping filters are reviewed in Secs. 4 and 5 as a preventative caution.

In Sec. 6, some pitfalls are revealed to occur in an early method that sought to tractably approximate the autocorrelation function of a general covariance stationary random process as a logical prelude to invoking shaping filter representation. An example from Sec. 2 conveniently serves as a counterexample to the method warned of in Sec. 6 by verifying its previously unacknowledged limited applicability. A summary of sorts is provided in Sec. 7.

Rather than invoke mere classical theorem/proof, a simple transparent story telling style is adopted throughout to facilitate a more concise breadth of topic coverage while still illustrating all concepts with concrete examples. What is unique here is the ordering and juxtaposition of the examples for clarity and pedagogical expediency. This particular strategy was pursued in order to warn a wider number of applications engineers than would otherwise be the case if a more terse discussion of the topics were utilized. In this same vein of attempting to increase the readership beyond just the specialist in this area, several alternative technical synonyms are provided to reduce confusion by using familiar language and standard jargon and ample supporting references and, moreover, pinpoint citations are supplied as backwards pointers for further elaboration or for crosscheck and detailed substantiation.

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2 Motivational Historical Context

The concept of capturing the requisite second order statistics by modeling wide-sense stationary random processes as white noise passed through so-called "shaping filters" has been employed for many years in the analysis of physical systems in such diverse areas as:

1. Handling of serially time-correlated noise in seeking to utilize a standard Wiener or Kalman filtering framework (that nominally expects only uncorrelated nonsingular white noise) by modeling the second order statistical structure of the time-correlated noise using state augmentation [1, pp. 133-135] which then unfortunately induces singularity of the resulting white measurement noise (whose singularity initially appears to be a barrier to successful Kalman or Wiener filter implementation but can be computationally handled by the techniques of [2], [3]);

2. Pre-whitening as a prelude to performing matched filtering when the additive noise is not originally white [4, pp. 142-143];

3. Modeling and tracking of randomly initiated target maneuvers as recently utilized in a novel application of missile guidance simulation [5], [6], [28];

4. Implementation of the Levinson-Durbin and Burg algorithms in modern spectral estimation [7].

All these techniques have been successfully applied even though several different random processes with radically different sample functions (synonyms: sample paths, trajectories, or realizations) can possess the same correlation function (viz., [8, ex. 3.10, p. 204]) and consequently the same power spectra (i.e., the same second order statistics).

While every positive definite function can be a valid covariance function of some process (in particular, of a Gaussian process [9, paragraph prior to Eq. 22]), only for stationary processes does a complete answer (i.e., consisting of both necessary and sufficient conditions) exist to the following: **QUESTION:** Under what conditions can the scalar random process $y(t)$ be represented as the output of a causal and causally invertible linear filter (having transfer function $H(\jmath\omega)$) with a white noise input, $w(t)$, where the scalar output spectral density $S_{yy}(\jmath\omega)$ is related to the scalar input spectral density $S_{ww}(\jmath\omega)$ by the familiar formula:

$$S_{yy}(\jmath\omega) = |H(\jmath\omega)|^2 S_{ww}(\jmath\omega) \tag{1}$$

where

$$S_{ww}(\jmath\omega) = q > 0 \tag{2}$$

and q is constant, representing the covariance intensity (power) of ideal stationary zero mean Gaussian white noise?

ANSWER: The Paley-Wiener condition:

$$\int_{-\infty}^{\infty} \frac{|\ln S_{yy}(\jmath\omega)|}{1+\omega^2}\, d\omega < \infty \tag{3}$$

is a necessary and sufficient condition [10, p. 114] that a known power spectrum of $y(t)$ can be represented using a shaping filter and white noise as in Eqs. 1 and 2. (Pointers to existing generalizations of the Paley-Wiener condition in several directions, including the important multi-dimensional case, are provided on [9, p. 265] but follow-up may likely be less satisfying than anticipated from the original advertisement. Also see [67], [68].) It is indicated in [11, p. 215] that Eq. 3 should be further augmented to require the presence of absolute values about $S_{yy}(\jmath\omega)$ and in addition, require that $|S_{yy}(\jmath\omega)|$ be square integrable on the open infinite interval $(-\infty, \infty)$ as a more proper statement of the necessary and sufficient Paley-Wiener condition depicted above in Eq. 3.

As derived in [10, pp. 111-114], for the damped harmonic oscillator described by the following time-invariant second order differential equation:

$$\ddot{y}(t) + 2\alpha\dot{y} + (\omega_o^2 + \alpha^2)y(t) = w(t) \tag{4}$$

with the dot above the variable signifying differentiation with respect to time and having scalar Gaussian white noise input w(t) with covariance intensity level as in Eq. 2. Use of the input-output formula of Eq. 1 for this situation yields:

$$S_{yy}(\jmath\omega) = \frac{q}{(\omega_o^2 + \alpha^2 - \omega^2)^2 + 4\alpha^2\omega^2} \tag{5}$$

Since the correlation function and the power spectrum are Fourier transform pairs [1], a table of transform pairs [12, No. 5, p. 25] demonstrates that the correlation function corresponding to Eq. 5 is

$$R_{yy}(\tau) = \frac{q}{4\alpha(\omega_o^2+\alpha^2)} e^{-\alpha|\tau|}\left[\cos(\omega_o\tau) + \frac{\alpha}{\omega_o}\sin(\omega_o|\tau|)\right] \tag{6}$$

This is a standard result. An even more startling result is that the correct output power spectrum and autocorrelation function for the undamped oscillator corresponding to Eq. 4 with

$$\alpha = 0 \tag{7}$$

cannot be obtained by taking the limit as a goes to zero in Eqs. 5 and 6, respectively; although the first limit exists as

$$\lim_{\alpha\to 0} S_{yy}(\jmath\omega) = \frac{q}{(\omega_o^2-\omega^2)^2} \tag{8}$$

the second does not since

$$\lim_{\alpha\to 0} R_{yy}(\tau) = \lim_{\alpha\to 0} \frac{q}{4\alpha(\omega_o^2+\alpha^2)} e^{-\alpha|\tau|}\left[\cos(\omega_o\tau) + \frac{\alpha}{\omega_o}\sin(\omega_o|\tau|)\right] \tag{9a}$$

$$= \infty + \frac{q}{4\omega_o^3}\sin(\omega_o|\tau|) = \infty, \tag{9b}$$

and ∞ here is notation for the limit diverging or not being defined for the situation being considered. While the finite portion of Eq. 9b is an even function and does not exceed the amplitude at $\tau = 0$ to conform to the properties that all proper autocorrelation functions must satisfy and the "power spectrum" of Eq. 8 is appropriately non-negative, this much-worn primrose path of seeking to utilize Eqs. 8 and 9 fails to yield the correct output power spectrum and correlation function for the undamped oscillator.

The apparent inconsistency stems from the fact that it is inappropriate to use Eq. 1 for a pure oscillator or for any linear system that is only marginally stable rather than strictly stable! To understand the nature of this limitation, one must recall how the familiar formula of Eq. 1 was originally derived from the fundamental superposition or convolution integral representing the input-output relation for a linear time-invariant system [13, p. 25] as:

$$Y_t(\omega) = \int_{-\infty}^{\infty} h(t-\tau)X_\tau(\omega)\, d\tau \tag{10}$$

where

$h(t)$ = deterministic impulse response of a time invariant linear filter;
$X_t(\omega)$ = sample function of a wide − sense stationary random process that is input to the linear system;
$Y_t(\omega)$ = sample function of a wide − sense stationary random process that is the output of the linear system.

The convolution in Eq. 10 can be interpreted on two different levels: either as an absolutely convergent Lebesgue integral for each different sample function of the $X_t(\omega)$ random process; or as a quadratic mean (q.m.) Riemann integral (i.e., limit in mean square sense [14, pp. 99-108],[15, pp. 66-70]). As stated in [16, p. 92] , to ensure the existence of Eq. 10 as an absolutely convergent Lebesgue integral, conditions must be imposed on the $X_t(\omega)$ process that are more restrictive than its being merely a second order process (thus the desired generality of Eq. 10 to all wide-sense stationary processes would be lost). Consequently, it is more convenient to interpret Eq. 10 as a q.m. integral, as is routinely done (whether or not it is explicitly recognized to be so by the user). The existence of Eq. 10 as a q.m. integral (i.e., Eq. 10 being well-defined and making sense as a mathematical entity) is assured if and only if

$$\int_{-\infty}^{\infty} h(t-\tau)h(t-u)R_{xx}(\tau-u)\, d\tau\, du = \int_{-\infty}^{\infty} |H(f)|^2 S_{xx}(f)\, df < \infty \tag{11}$$

and $R_{xx}(\cdot)$ is both L^1 and *continuous* (see [16, pp. 92-93 prior to Eq. 5.17]. Once Eq. 10 has a valid interpretation, it can be multiplied times itself (for a different time index and dummy variable of integration) and expectation taken throughout to yield

$$R_{yy}(t,s) \triangleq E[Y_t(\omega)Y_s(\omega)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)h(s-u)R_{xx}(\tau-u)\, d\tau\, du \tag{12a}$$

$$= \int_{-\infty}^{\infty} e^{\jmath 2\pi f(t-s)}|H(f)|^2 S_{xx}(f)\, df \tag{12b}$$

[1] An original observation is that the time domain expression in Item 4 of [12] should be multiplied by 2, otherwise the sums of Item Nos. 5 and 6 are inconsistent with the Item 4.

where an interchange in the order of expectation and integration has been performed in Eq. 12a (c.f., [8, p. 82]). Since the operation of expectation is itself an integral over a sample space with respect to a probability measure (or some such equivalent notion using the monotonely increasing cumulative distribution function in a Riemann-Stieltjes integral or alternatively using the probability density function itself, if it exists), the inter-change of expectation and integration that occurs in Eq. 12a is actually an integral-integral interchange And, as such, is validated in [16, p. 93] under the sufficient (but stronger than necessary) assumption that

$$\int_{-\infty}^{\infty} |h(\tau)|^2 \, d\tau < \infty \tag{13}$$

(this condition also being sufficient but not necessary that $h(t)$ be Fourier transformable [17, p. 31]). Therefore, if the condition of Eq. 13 is satisfied and the condition of Eq. 11 is satisfied, then Eq. 12 holds and both sides can be validly inverse Fourier transformed to yield Eq. 1.

The following condition, reminiscent in form to the condition of Eq. 13, being

$$\int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty \tag{14}$$

is equivalent to the linear system being bounded input-bounded output stable in the strict sense (i.e., disallowing marginally stable systems with poles on the $\jmath\omega$-axis), as proved in a Theorem provided in [17, pp. 28-29]. However, for the linear undamped oscillator of Eq. 4, with parameter as in Eq. 7, both the conditions of Eqs. 13 and 14 are violated [2] since, respectively,

$$\int_{-\infty}^{\infty} |A\sin(\omega_o\tau) + B\cos(\omega_o\tau)| \, d\tau = \int_{-\infty}^{\infty} |\sqrt{A^2+B^2}\sin(\omega_o\tau + \arctan\{\frac{B}{A}\})| \, d\tau = \infty \tag{15}$$

and

$$\int_{-\infty}^{\infty} |A\sin(\omega_o\tau) + B\cos(\omega_o\tau)|^2 \, d\tau = \int_{-\infty}^{\infty} \frac{(A^2+B^2)}{2}[1-\cos(2\omega_o\tau + 2\arctan\{\frac{B}{A}\})] \, d\tau = \infty \tag{16}$$

Consequently, the condition of Eq. 13 does not hold for the undamped oscillator of Eqs. 4 and 7, thus the interchange of integration in Eq. 13 is not validated and its corresponding inverse Fourier transform, as Eq. 1 is not validated as well.

Frequently, engineers with a practical bent are of the opinion that the order of integration can be routinely interchanged whenever both versions of the alternative integral interchange exist. That this is not strictly the case is demonstrated via the counterexample in Appendix A.

The exact correlation function for a marginally stable undamped oscillator forced by a Gaussian white noise process has already been worked out in detail using two different but rigorous approaches in [14, pp. 158-161]. One approach utilizes the white noise directly, while the other approach (acknowledging that white noise is a mathematically inconsistent albeit useful fiction) instead resorts to dealing with the more rigorously consistent underlying Brownian motion or Wiener process. The correct correlation function, corroborated by both approaches in [14], is

$$R_{yy}(t,s) = \frac{q}{\omega_o^2}\left[t\cos(\omega_o[t-s]) - \frac{1}{2\omega_o}\sin(\omega_o[t-s]) - \frac{1}{2\omega_o}\sin(\omega_o[t+s])\right] \tag{17}$$

for $s \leq t$ and vice-versa by interchanging t and s in Eq. 17 for $t > s$. Since the first and last terms on the right side of Eq. 14 are not a function of only the time difference $(t-s)$, the correlation function of Eq. 17 represents a nonstationary process. This nonstationarity of the undamped oscillator is further acknowledged in [19, ex. 45, pp. 167-168]. This conclusion of nonstationarity is consistent with what might be reasonably expected from simple physical intuition for a linear system having a deterministic impulse response consisting of a pure undamped oscillator being excited by white noise having the same level of energy in all frequencies, and so capable of stimulating (as with deterministic resonance phenomena) the natural frequency of the oscillator. Occurrences of this type of stochastic resonance phenomenon have reportedly been encountered by TASC [1] in early 1970's navigation applications.

[2] There are in fact some functions that are $L^1[-\infty,\infty]$ but not $L^2[-\infty,\infty]$ and vice-versa, as explicitly demonstrated in [18, p. 17, ex. 3]. However, over a finite interval of integration (i.e., where both upper and lower limits in the associated integral have magnitudes less than infinity and are specified explicitly), being $L^2[a,b]$ ensures also being $L^1[a,b]$ by virtue of the Cauchy-Schwarz inequality.

3 On Ergodicity, Stationarity, and Mixing

On the other hand, a well-known random process that has sinusoidal sample functions of differing initial phase is:

$$y(t) = A\sin(\omega_o t + \theta) \tag{18}$$

where t is continuous time, A and ω_o are deterministic and θ is a random variable with theta uniformly distributed in the interval $[0, 2\pi]$ as

$$p_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & \text{for } 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The random process of Eq. 18 is known to be wide-sense stationary [20, p. 286] since the correlation is a function of time difference only

$$R_{yy}(t, s) = \frac{A^2}{2}\cos(\omega_o[t - s]) \tag{20}$$

and the mean is a constant

$$E[y(t)] = 0 \tag{21}$$

moreover, the first order probability density function for $y(t)$ in Eq. 18 exists [20, p. 272] as

$$p_y(y, t) = \begin{cases} \frac{1}{\pi\sqrt{1-y^2}} & \text{for } |y| \leq 1, \\ 0 & \text{for } |y| > 1, \end{cases} \tag{22}$$

which is independent of the particular time t, therefore, indicating that $y(t)$ is stationary. The above process is obviously non-Gaussian as can be inferred from Eq. 22. Since $y(t)$ of Eq. 8 is non-Gaussian, it cannot be represented as the output of a linear system driven by Gaussian white noise; otherwise the output process y(t) would also have to be Gaussian (i.e., Gaussian into a linear system, consequently has Gaussian out)! In addition to being stationary, the $y(t)$ process of Eq. 1 is ergodic [20, p. 290] (i.e., time-averages and ensemble averages are equal); and as is evident for the first two moments from the demonstration below that the time averages of Eq. 1 as

$$< y(t) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\sin(\omega_o t + \theta)dt = 0 \tag{23}$$

and

$$< y(t)y(t+\tau) >= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A^2 \sin(\omega_o t + \theta)\sin(\omega_o[t+\tau] + \theta)\ dt = \frac{A^2}{2}\cos(\omega_o \tau) \tag{24}$$

correspond to the ensemble averages of Eqs. 21 and 20, respectively. For general second order random processes, there are explicit necessary and sufficient conditions in terms of fourth order moments [21, p. 330] for establishing ergodicity of the autocorrelation function; however, these conditions simplify to a test involving only the ensemble autocorrelation function for the special but prevalent case of a zero mean Gaussian process (since all even higher moments of Gaussian processes reduce to functions of just the second only). Ergodicity implies stationarity, but more recent assumptions on a process as being "mixing" [22], [23, p. 488], [24], [25] (as a form of asymptotic independence) implies ergodicity (but not vice-versa).

Unfortunately, a recent Theorem 6 on p. 786 of [26] refers to matrices having elements that are random processes, being both *phi*-mixing and yet also <u>non</u>stationary, an apparent contradiction to historical notions, and a definite contradiction to the well-established nesting of properties, as provided in [27, p. 71].) The topic of this well known example of Eqs. 18, 19, 20 was raised here as a prelude to the discussion in Sec. 6, where this same random process is used as a counterexample to another fairly prevalent misconception.

4 An Extension with Potential Utility in Simulating Randomly Initiated Missile Maneuvers

The concept of a shaping filter excited by white noise is extended in [5], [6], [28] to apply to the statistical representation of disturbances with known form but random starting time. The contrivance that is exploited in [5] is that second order statistics of a process (such as the mean-squared value being one such of many possibilities) remain invariant if the original process is replaced by another having the same autocorrelation function. The type of input considered as constituting the replacement is of the form

$$y_t(\omega) = h(t - T(\omega)), \tag{25}$$

where

$h(\cdot)$ = a disturbing function of known form which vanishes for negative values of the argument;
$T(\omega)$ = a random starting time of the disturbance;
$p_T(t)$ = known probability density function (pdf) for the random starting time.

Using Eq. 25 and the pdf of the random starting time, the fundamental theorem of expectation [8, p. 85] yields the following expression for the autocorrelation as

$$R_{yy}(t,s) \triangleq E[y(t)y(s)] = \int_{-\infty}^{\infty} h(t-\tau)h(s-T)p_T(T)\, dT \tag{26a}$$

$$= \int_{-\infty}^{min(t,s)} h(t-T)h(s-T)p_T(T)\, dT. \tag{26b}$$

However, if $y_t(\omega)$ here is assumed to be the output of a shaping filter or impulse response $h(\cdot)$ with possibly non-stationary white noise input, then the expression for the output autocorrelation is

$$R_{yy}(t,s) = \int_{-\infty}^{t} h(t-\tau_1)\int_{-\infty}^{s} h(s-\tau_2)q(\tau_1)\delta(\tau_1-\tau_2)\, d\tau_1\, d\tau_2 \tag{27a}$$

$$= \int_{-\infty}^{t} h(t-\tau)h(s-\tau)q(\tau)\, d\tau, \tag{27b}$$

where the upper limit occurs in both Eqs. 26b and 27b as a result of $h(\cdot)$ being assumed to vanish for negative arguments. As developed in [5], [6], the process described by Eq. 26 and the process described by Eq. 27 have equivalent second order statistics if the time-varying covariance intensity level of the nonstationary white noise relates to the pdf as

$$q(t) = p_T(t). \tag{28}$$

Unfortunately, a problem arises in [5], [6], [28] in the application of this technique of replacement by a process with equivalent second order statistics. Please notice from Table 1 of [5, p. 3] and in [6, p. 153], also depicted here, that Item D consists of an undamped sinusoidal sample function and an associated block diagram of an undamped second order linear time-invariant system that is prescribed as the appropriate shaping filter to produce the desired matching output correlation function. As a consequence of the inconsistency of this view with the previous rigorous discussion in Sec. 2 of the appropriate nonstationary correlation function (Eq. 17) as the output of an undamped second order system forced by even stationary white noise and the non-Gaussianess of the process of Eq. 18 (as initialized by Eq. 19), it is immediately recognized that Item D in Table 1 of [5], [6], [28] should be omitted as incorrect to avoid erroneous conclusions while the remaining Items C to G in Table 1 of these references appear to be appropriate without modification, but Items A and B are also incorrect (e.g., integrated Gaussian white noise is a random walk [1, p. 83] which is also well-known to be nonstationary).

In assessing missile intercept capabilities in Table 2 [28, Fig. 2], [5], the improper Item D of Table 1 is unfortunately utilized as a one-dimensional representation of a "barrel roll" executed by a target during an evasive maneuver (for the linear case of no gravity saturation). Also in Table 2 from [28], the improper Item D is also utilized twice (in a two term Fourier series expansion [5, p. 6]) in representing a target executing a "vertical S" maneuver in which the target attempts to pull maximum g's alternatively upward then downward. Both of the above representations of evasive target maneuvers in simulations in both [6] and [28], as seen from Table 2, appear to suffer from problems of utilizing an improper shaping filter, as already discussed above since the output would be nonstationary for these cases and, moreover, Eqs. 1 and 12 would not be applicable. Recall that finite time snippets of non-stationary processes are again non-stationary. The multiplication of the nonstationary marginally stable system's output by a single finite duration unit pulse gate in the time domain is equivalent to convolution in the frequency domain, which would smear out and attenuate the effect. However, use of periodic maneuvers would correspond to multiplying the marginally stable system's output by a train of finite duration unit pulses and the corresponding convolution in the frequency domain would not attenuate the effect.

However, a rigorously defined technique has been more recently proposed as [29] for validly obtaining the output ensemble average, even if the linear system under consideration is time-varying; however, the input must be of an especially restrictive benign form (such that it is both sinusoidally periodic and covariance stationary) to be able to use the approach of [29] in applications. The maneuvers of interest apparently do not satisfy this constraint.

Table 1. Shaping filters for some disturbances (from [5],[6]).

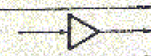
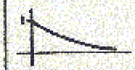
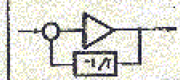
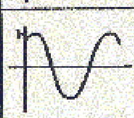
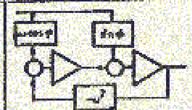
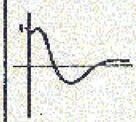
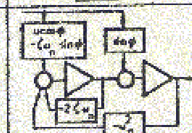
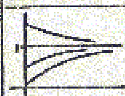
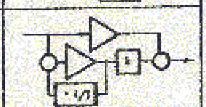
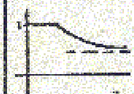
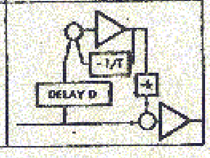
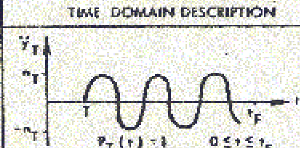
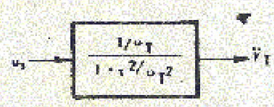
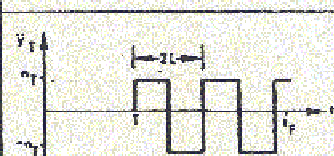
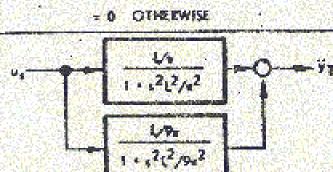
	TYPE	h(t) (t > 0)	IMPULSE RESPONSE CURVE	STATE DIFFERENTIAL EQUATIONS	BLOCK DIAGRAM
A	STEP	1		$\dot{x} = w$	
B	RAMP	t		$\dot{x} = x_1$ $\dot{x}_1 = w$	
C	EXPONENTIAL	$e^{-t/T}$		$\dot{x} = \frac{-1}{T} x + w$	
D	SINUSOID (ARBITRARY PHASE)	$\sin(\omega t + \phi)$		$\dot{x} = x_1 + (\sin\phi) w$ $\dot{x}_1 = -\omega^2 x + (\omega \cos\phi) w$	
E	DAMPED SINUSOID	$e^{-\zeta\omega_n t} \sin(\omega t + \phi)$ $(0 < \zeta < 1,$ $\omega = \omega_n \sqrt{1 - \zeta^2})$		$\dot{x} = x_1 + (\sin\phi) w$ $\dot{x}_1 = -\omega_n^2 x - 2\zeta\omega_n x_1 + (\omega\cos\phi - \zeta\omega_n \sin\phi) w$	
F	STEP PLUS EXPONENTIAL	$1 + k e^{-t/T}$		$x = x_1 + k x_2$ $\dot{x}_1 = w$ $\dot{x}_2 = \frac{-1}{T} x_2 + w$	
G	FUNCTION WITH PURE TIME DELAY	$1 - k u(t-D) \left[1 - e^{-(t-D)/T}\right]$		$\dot{x} = -k x_1 + w$ $\dot{x}_1(t) = \frac{-1}{T} x_1(t) + w(t-D)$	

Table 2. Shaping filter equivalents for various maneuver policies (from [28]).

NAME	TIME DOMAIN DESCRIPTION	SHAPING FILTER EQUIVALENT
BARREL ROLL WITH INITIATION TIME UNIFORMLY DISTRIBUTED OVER FLIGHT TIME	 $P_T(\tau) = \frac{1}{T_F}$ $0 \le \tau \le t_F$ $= 0$ OTHERWISE ω_T - FREQUENCY OF SINUSOID	 u_s = WHITE NOISE WITH POWER SPECTRAL DENSITY $q_s(\tau)$ $q_s(\tau) = n_T^2/t_F$ ft²/s⁵ $0 \le \tau \le t_F$ $= 0$ OTHEREWISE
VERTICAL - S WITH INITIATION TIME UNIFORMLY DISTRIBUTED OVER FLIGHT TIME	 $P_T(\tau) = \frac{1}{t_F}$ $0 \le \tau \le t_F$ $= 0$ OTHERWISE L = HALF PERIOD OF MANEUVER	 u_s = WHITE NOISE WITH POWER SPECTRAL DENSITY $q_s(\tau)$ $q_s(\tau) = \frac{16 n_T^2}{\pi^2 t_F}$ ft²/s⁵ $0 \le \tau \le t_F$ $= 0$ OTHERWISE

5 Other Erroneous Applications of Shaping Filters in Inherently Nonstationary Situations

The misapplication of shaping filters for second order undamped systems is just symptomatic of an unfortunate but prevalent general trend toward improper usage as can sometimes occur in special studies as discussed in [31] involving standard linear error models of navigation systems [30] (where both 24 hour earth-rate oscillations and $2\pi\sqrt{R/g} = 84$ minute Schuler oscillations occur, with g being the acceleration of gravity and R being the mean radius of the earth). These same errors of improper usage (as described throughout this paper) have assumed different guises in the past such as in an unnamed researcher's attempt to quantify the mean squared error of a marginally stable system by utilizing Parseval's identity [20, p. 15] in conjunction with Eq. 12 and the condition of Eq. 2 as

$$E[y^2(t)] = R_{yy}(t,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t-\tau)h(t-u)q\delta(\tau-u)\,d\tau\,du \quad (29a)$$

$$= \int_{-\infty}^{\infty} |h(t-\tau)|^2 q\,d\tau \quad (29b)$$

$$= q\int_{-\infty}^{\infty} |H(f)|^2\,df. \quad (29c)$$

Tables do exist (viz., [32]-[34]) to allow the convenient evaluation of Eq. 29c without having to explicitly perform integration for the standard case of $H(f)$ being a rational function (i.e., the ratio of two polynomials in f). (Ref. [33, Appendix] appears to provide the most extensive table and its derivation [33, Sec. 7.9], but [34] indicates proper generalization to any order of numerator/denominator. Both [33] and [34] are more convenient to use than [32] since the numerator in the integrand of Eq. 28c need not be factored in order to use them.) The problem, as has often been cautioned against in the past, is the unsubstantiated utilization of the tables of [32]-[34] to evaluate Eq. 29c for marginally stable systems since following this procedure can yield (proper appearing) finite answers, with no warning that Eq. 29 has been misapplied to a marginally stable nonstationary system (since reasonable looking but incorrect "answers" are obtained from this erroneous approach). The correct answer is that the integral blows up and is infinite in the marginally stable case. These earlier tables, mentioned above, are perhaps rendered obsolete by the simple recursive algorithms and short 23 line FORTRAN implementation code offered in [41, Chap. 5] for performing the requisite evaluations. Karl Astrom credits J. Nekolny [41, p. 157] for originating the recursive evaluation of these integrals in 1957. An interesting connection is that the pre-1970 derivations of [41] in this regard have the same form as the order-recursive relation in the Levinson-Durbin algorithm [7, pp. 82-85, 99-10] that also involve autoregressive (AR) systems with associated poles that are guaranteed to be within the unit circle and therefore are stable at each recursion by virtue of the corresponding reflection coefficients having magnitudes that are less than unity. Actual simple proofs of the stability that ensues are provided in [41, Chapt. 5].

Another guise this problem has assumed in the past appears in [35, pp. 39-41], where for a time invariant linear system of the form

$$x(t) = Fx(t) + Gu(t) \quad (30)$$

$$y(t) = Hx(t), \quad (31)$$

where

$u(t)$ = zero mean Gaussian white process noise (ref. [35] instead utilizes the equivalent but more mathematically rigorous Brownian Motion representation of the process noise);

$Q = Q^T$ = covariance level (being a positive definite matrix) of the above white process noise, $u(t)$;

then the corresponding second-order statistics may be represented in the time-domain and in the frequency-domain, respectively, by the following matrix concatenations:

$$R_{yy}(t,\tau) = \begin{cases} He^{F(t-\tau)}PH^T & \text{for } t > \tau \\ HPe^{F^T(\tau-t)}H^T & \text{for } \tau \geq t \end{cases}, \quad (32a)$$

and

$$S_{yy}(\jmath\omega) = H(\jmath\omega I - F)^{-1}GQG^T(-\jmath\omega I - F^T)^{-1}H^T \quad (33a)$$

$$= H(\jmath\omega I - F)^{-1}PH^T + HP(-\jmath\omega I - F^T)^{-1}H^T. \quad (33b)$$

Of course corresponding, straightforward, equivalent discrete-time versions of Eqs. 32 and 33 above also exist. In Eqs. 32 and 33 here, the P is required to be the positive semi-definite solution to the following steady-state Lyapunov equation

$$FP + PF^T = -GQG^T, \tag{34}$$

and

$$E[x(t)x^T(t)] = P. \tag{35}$$

However, ref. [35] implicitly considers only the case when a positive semi-definite solution to Eq. 34 is assumed to exist rather than invoke the well-known explicit conditions (discussed in [42]) when solutions do exist. This absence of conditions on Eq. 34 in [35] is somewhat paradoxical since [35, pp. 65-80] provides one of the most succinct considerations of existence and uniqueness of solutions of the related Riccati equation (e.g., in order to recursively obtain the steady-state solution of the Riccati equation via a quadratically convergent or even geometrically convergent [36] Newton-Ralphson iteration procedure, the algorithm of [37], [38] repeatedly obtains a steady-state solution of a Lyapunov equation, as discussed in [39, pp. 252-254] with related comments in [40]). The recursive multi-channel generalization or extension by Nuttall of the Maximum Entropy Method for spectral estimation also requires solution of an algebraic matrix Lyapunov equation of the form of Eq. 34 (c.f., fourth unnumbered equation from the top of Sec. 3 in [51]) on each iteration of the recursion. Because [35] enjoys such renown for being exacting on topics relating to estimation, the presentation in [35] of Eqs. 32, 33, and 34 appearing without the explicit cautionary conditions summarized in [42] (for when a positive semi-definite solution to Eq. 34 exists) was probably widely disseminated. Within the Maximum Entropy Method (MEM) of power spectrum estimation, utilization is made of the relationship of Eq. 1 or, more exactly, of its multidimensional or multi-channel discrete-time generalization (reminiscent of a variant of Eq. 33a) as specialized to p^{th} order autoregressive AR processes (e.g., [1, pp. 90-95]). The alternative versions of multidimensional MEM ([47]-[51]) utilize a multidimensional generalization of scalar AR processes. Finally, within some of these alternative approaches to generalize the scalar MEM to be applicable in the multi-channel case, recursive algorithms for the autoregressive coefficients are encountered. Here again it is important that the roots of the AR process correspond to a strictly stable system otherwise Eqs. 1 and 33 are meaningless and the AR process will be nonstationary. As indicated in [51], [63], both the Levinson-Wiggins-Robinson (LWR) and Nuttall multidimensional techniques contain a theoretical proof of the stability of the resulting AR process, but unfortunately the effects of performing computations with finite word length arithmetic can cause some roots to not be stable. In these unfortunate situations, the underlying Eqs. 1 or 33 are not valid and this can cause problems during successive iterations of these recursive techniques. The main point here is that some computer programs implement a multichannel MEM routinely by implicitly utilizing Eq. 1 on each iteration, however, roots of the AR process associated with that particular iteration can be unstable (as induced computationally by numerical round-off) so that Eq. 1, as utilized, is no longer valid.

6 A Pitfall in One Procedure Intended to Tractably Approximate Stationary Autocorrelation Functions

In [45, Appendix D], an ostensibly general approach is suggested for approximating the 1-D power spectral density of a scalar stationary process $S(\jmath\omega)$ in terms of rational functions of prescribed form. It is further suggested in [45] that for convenience, the technique should be applied to the autocorrelation function $R(\tau)$ in an equivalent manner in the time-domain using an approximating finite series for $R(\tau)$ of the form [45, Eq. D-1]:

$$R(\tau) \approx \sum_{k=1}^{n} A_k e^{-c_k|\tau|}, \tag{36}$$

where the exponent constants c_k in Eq. 36 are neither imaginary nor complex [but purely real], making it impossible to simply relate Eq. 36 to a sine and cosine series via Euler's identity. The procedure of [45] assertedly consists of further constructing an orthonormal set $\{\Psi_k(\tau)\}_{k=1}^{\infty}$ over the entire real line (the domain of τ) from the original set of independent functions $\{e^{-c_k|\tau|}\}_{k=1}^{\infty}$ via a standard Gram-Schmidt ortho-normalization, then determining the appropriate coefficients B_k in the alternative approximation of the correlation function of interest as

$$R(\tau) \approx \sum_{k=1}^{n} B_k \Psi_k(\tau), \tag{37}$$

where the convergence indicated is asserted to be "in mean square" rather than "point-wise" which is consistent with acknowledging adverse possibilities such as Gibbs' phenomenon occurring using these usual techniques for orthonormal sets of functions. A cautionary observation being made here is that, unlike what is asserted [45, p. 384, second sentence], neither the sequence $\{e^{-c_k|\tau|}\}_{k=1}^{\infty}$ nor its corresponding orthonormalized sequence $\{\Psi_k(\tau)\}_{k=1}^{\infty}$ is *complete* [61, Sec. 6.02], (i.e., complete in the sense that the only function that is orthogonal to every element in the sequence is the null or zero function; the term "completeness" being a concept that originated in connection with Fourier series since for an arbitrary piece-wise continuous function that is neither even nor odd, it is well known that neither the infinite sine series nor the infinite cosine series alone are complete but together they are complete by spanning the whole space of continuous functions (this definition of completeness is in contradistinction to other prevalent notions of completeness in mathematics such as "completeness of measures" [44, p. 27], or *Banach* and *Hilbert spaces* being "complete" in the sense that all Cauchy sequences only converge to points within the space or ground set [44, p. 76]). Analogously, the infinitude of both the 1^{st} and 2^{nd} kind of solutions of standard Sturm-Liouville problems [43], [61, Chapt.6], [62, Chapt. 22], [66] are needed in order to span the entire solution function space and use of the infinite collection of just one kind by itself is insufficient to do so. If the set of functions $\{e^{-c_k|\tau|}\}_{k=1}^{\infty}$ were cosine functions instead, then this collection would have sufficed since the correlation function of a stationary process is an even function.

To illustrate that the procedure recommended in [45, App. D] is not applicable, in general, without further yet to be enumerated restrictions imposed, please consider application of Step 1 on [45, p. 384] to the stationary correlation function of Eqs. 20 or 24. Step 1 suggests "subtracting a constant from $R_{yy}(\tau)$ so that the difference approaches zero (in the limit) as $\tau \to \infty$", but accomplishment of this directive is impossible to carry out for $R_{yy}(\tau)$ in Eq. 20 since it is periodic in *tau*, and no alternative recourse for properly handling this non-pathological problem is availed in [45] so apparently the procedure offered in [45] is not applicable to even this common stationary correlation function. Fortunately, the entirety of Chapter V of [46] is devoted to the following four alternative approaches for correctly approximating stationary spectral densities in the frequency domain with rational functions via:

- interpolation;
- approximation based on a Fourier series expansion;
- approximation based on a Laguerre series expansion;
- approximation of the logarithmic curve of the power spectral density by intersecting straight line segments.

However, the attainment of adequate frequency domain or time domain representations of power spectra and correlation functions, respectively, may be more of a moot point now with the advent of several modern techniques to provide shaping filter representations as an output [64]. By operating directly on the measured data, without first forming sample correlation functions as an intermediate step [65], variants of the Maximum Entropy Method (MEM) can provide as output a shaping filter consisting of all-poles with autoregressive (AR) coefficients completely specified.

7 Summary/Conclusions

Standard input/output relations for linear system autocorrelation functions and power spectra representation were reviewed in Sec. 2 and limitations or restrictions on their applicability were revealed and emphasized. A low dimensional second-order example was used to illustrate the nature of the problem. Some observations were made in Sec. 3 on the interrelationships of ergodicity, stationarity, and proper phi-mixing. Sec. 4 discussed an apparent problem in applying a novel somewhat recent technique since the shaping filter in the application corresponds to an undamped oscillatory system of the form cautioned against in Sec. 2 as being neither stable nor stationary. Other instances of shaping filters or other related equivalent notions being attempted in inherently nonstationary situations are recounted in Sec. 5 as a warning to analysts. A few pitfalls are revealed in Sec. 6 to exist in an early method that sought to tractably approximate the autocorrelation function of a general covariance stationary process. For additional original clarifying counterexamples that we have provided in a number of other areas of modern estimation and control, please see [52]-[59], [68]-[72].

A A Surprise in Interchanging the Order of Integration

Consider evaluation of the following innocent looking integral:

$$I \triangleq \int_0^1 \int_1^{\infty} (e^{-xy} - 2e^{-2xy})\, dx\, dy. \tag{38}$$

In order to demonstrate that the order of integration cannot be validly interchanged for this integral, a result concerning another related integral is useful. The related integral to be evaluated is:

$$\int_0^\infty \frac{(e^{-t} - 2e^{-2t})}{t}\, dt. \tag{39}$$

Evaluation of Eq. 39 is accomplished easily by exploiting properties of unilateral Laplace transforms. The numerator of the integrand of Eq. 39 consists of the following function of exponentials:

$$f(t) = (e^{-t} - 2e^{-2t}), \tag{40}$$

which has the obvious Laplace transform:

$$F(s) = \frac{1}{s+1} - \frac{1}{s+2}. \tag{41}$$

By a well known property of the Laplace Transform [43, p. 53]:

$$\begin{aligned}
\int_0^\infty [f(t)\frac{e^{-st}}{t}]\, dt &= \int_s^\infty [\frac{1}{u+1} - \frac{1}{u+2}]\, du && \text{(42a)}\\
&= [\ln(u+1) - \ln(u+2)]\,|_s^\infty && \text{(42b)}\\
&= \ln|\frac{(u+1)}{(u+2)}||_s^\infty = \lim_{u\to\infty} \ln|\frac{(u+1)}{(u+2)}| + \ln|\frac{(s+2)}{(s+1)}| && \text{(42c)}\\
&= \ln|\lim_{u\to\infty} \frac{(1+\frac{1}{u})}{(1+\frac{2}{u})}| + \ln|\frac{(s+2)}{(s+1)}| && \text{(42d)}\\
&= \ln(1) + \ln|\frac{(s+2)}{(s+1)}| = \ln|\frac{(s+2)}{(s+1)}| && \text{(42e)}
\end{aligned}$$

for all s with real part greater or equal to the abscissa of convergence, which in this example can be taken to be $-1/2$ since all the poles of $F(s)$ are to the left of $-1/2$. In particular, for the special case of

$$s = \frac{1}{n}, \tag{43}$$

the integral of Eq. 42 simplifies as

$$\int_0^\infty [f(t)\frac{e^{-t/n}}{t}]\, dt = \ln \frac{(\frac{1}{n}) + 2}{(\frac{1}{n}) + 1} \tag{44}$$

for any positive integer n. Using this intermediate result, the objective of evaluating the auxiliary integral of Eq. 39 can now be taken to completion as:

$$\begin{aligned}
\int_0^\infty [\frac{f(t)}{t}]\, dt &= \int_0^\infty \lim_{n\to\infty} [f(t)\frac{e^{-t/n}}{t}]\, dt && \text{(45a)}\\
&= \lim_{n\to\infty} \int_0^\infty [f(t)\frac{e^{-t/n}}{t}]\, dt && \text{(45b)}\\
&= \lim_{n\to\infty} \ln[\frac{(\frac{1}{n}) + 2}{(\frac{1}{n}) + 1}] && \text{(45c)}\\
&= \ln[\lim_{n\to\infty} \frac{(\frac{1}{n}) + 2}{(\frac{1}{n}) + 1}] = \ln[2] \neq 0, && \text{(45d)}
\end{aligned}$$

where the interchange in the order of limit taking and integration is allowed in going from Eqs. 45a to 45b via the Lebesgue monotone convergence theorem [44, p. 21] and the interchange of limit taking and taking the natural logarithm in the last step in Eqs. 45d and in going from Eq. 45c to 45d is permissible since the function $\ln(\cdot)$ is a continuous function on $(0, \infty)$ and every continuous function is sequentially continuous [60, p. 77, Theorem 5.2.6] (i.e., its value at the limit of the sequence $\{\frac{1}{n}\}_{n=1}^\infty$ is the same as the limit of the term by term evaluations of the sequence).

Now that the auxiliary integrals of Eqs. 39 and 45 have been evaluated and seen to be non-zero via a mathematically rigorous path, verification can now proceed in demonstrating that the order of integration of the original integral of Eq. 38 cannot be validly interchanged.

The verification of the above assertion will be proved by contradiction. First, assume that the order can be interchanged so that:

$$I \triangleq \int_0^1 \int_1^\infty (e^{-xy} - 2e^{-2xy})\, dx\, dy \tag{46a}$$

$$= \int_1^\infty \int_0^1 (e^{-xy} - 2e^{-2xy})\, dy\, dx. \tag{46b}$$

Now the integral in Eq. 46a can be iteratively integrated once to yield:

$$\int_0^1 \int_1^\infty (e^{-xy} - 2e^{-2xy})\, dx\, dy = \int_0^1 [\frac{(e^{-xy} - 2e^{-2xy})}{y}]|_1^\infty\, dy \tag{47a}$$

$$= \int_0^1 [\frac{(e^{-y} - 2e^{-2y})}{y}]\, dy = \int_0^1 [\frac{(e^{-t} - 2e^{-2t})}{t}]\, dt. \tag{47b}$$

Similarly, the integral in Eq. 46b can be iteratively integrated once to yield:

$$\int_1^\infty \int_0^1 (e^{-xy} - 2e^{-2xy})\, dy\, dx = \int_1^\infty [\frac{(-e^{-xy} + 2e^{-2xy})}{x}]|_0^1\, dx \tag{48a}$$

$$= \int_1^\infty [\frac{(-e^{-x} + 2e^{-2x})}{x}]\, dx = \int_1^\infty [\frac{(-e^{-t} + 2e^{-2t})}{t}]\, dt. \tag{48b}$$

Since the two double integrals have been assumed to be equal in Eq. 46, quantities that each side is equivalent to as Eqs. 47b and 47b, respectively, should also be equal to each other as

$$\int_0^1 [\frac{(e^{-t} - 2e^{-2t})}{t}]\, dt = \int_1^\infty [\frac{(-e^{-t} + 2e^{-2t})}{t}]\, dt \tag{49}$$

Algebraically manipulating the integral equality of Eq. 49 so that both are on the same side of the equality sign yields:

$$0 = \int_0^1 [\frac{(e^{-t} - 2e^{-2t})}{t}]\, dt + \int_1^\infty [\frac{(e^{-t} + 2e^{-2t})}{t}]\, dt = \int_0^\infty [\frac{(e^{-t} - 2e^{-2t})}{t}]\, dt, \tag{50}$$

however, the above result equaling zero contradicts the auxiliary but rigorously derived result of this same integral on the right hand side of Eq. 50 being nonzero in Eq. 45d. Therefore, the results of the two iterated integrations in Eq. 46 are not equal (e.g., Eq. 46a $\neq$ Eq. 46b) even though each exists separately (and each is finite). QED

In order to always get correct answers, analysts must use creativity to proceed only along valid substantiated paths if rigor is to be claimed. A problem is that frequently analyst's managers may not understand why such care is needed (or care). This paper demonstrates why.

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References

[1] Gelb, A. (Ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, MA, 1974.

[2] Bryson, A. E., Jr. and Henrikson, L. J., "Estimation using sampled data containing sequentially correlated noise," *AIAA Journal of Spacecraft and Rockets*, Vol. 5, No. 6, pp. 662-665, Jun. 1968.

[3] Moylan, P. J., "A note on Kalman-Bucy filters with zero measurement noise," *IEEE Trans. on Automatic Control*, Vol. 19, No. 3, pp. 263-264, Jun. 1974.

[4] Hancock, J. C., *The Principles of Communication theory* McGraw-Hill, NY, 1961.

[5] Fitzgerald, R. J. and Zarchan, P., “Shaping filters for randomly initiated target maneuvers,” presented at *1978 AIAA Guidance and Control Conference*, Palo Alto, CA, 7-9 Aug. 1978.

[6] Fitzgerald, R. J., “Shaping filters for disturbances with random starting times,” *AIAA Journal of Guidance and Control*, Vol. 2, No. 2, Mar.-Dec. 1979.

[7] Proakis, J. G., *Digital Communications*, McGraw-Hill, NY, 1983.

[8] Wozencraft, J. M., and Jacobs, I. M., *Principles of Communication Engineering*, John Wiley, NY, 1965.

[9] Wong, E., “Recent progress in stochastic processes-a survey,” *IEEE Transactions on Information Theory*, Vol. 19, No. 3, pp. 262-275, May 1973.

[10] Parzen, E., *Stochastic Processes*, Holden-Day, San Francisco, CA, 3rd Printing, 1967.

[11] Papoulis, A., *The Fourier Integral*, McGraw-Hill, NY, 1962.

[12] Stratonovich, R. L., *Topics in the Theory of Random Noise*, Vol. I, Translated by Richard Silverman, Gordon and Breach, 2nd Printing, NY, 1967.

[13] DeRusso, P. M., Roy, R. J., and Close, C. M., *State Variables for Engineers*, John Wiley, NY, 1967.

[14] Soong, T. T., *Random Differential Equations in Science and Engineering*, Academic Press, NY, 1973.

[15] Jazwinski, A. H., *Stochastic Processes and Filtering Theory*, Academic Press, NY, 1970.

[16] Wong, E., *Stochastic Processes in Information and Dynamical Systems*, McGraw-Hill, NY, 1971.

[17] Sakrison, D. J., *Communication Theory: Transmission of Waveforms and Digital Information*, John Wiley, NY, 1968.

[18] Desoer, C. A. and Vidyasagar, M., *Feedback Systems: Input-Output Properties*, Academic Press, NY, 1975.

[19] Maybeck, P. S., *Stochastic Models, Estimation, and Control*, Academic Press, NY, 1979.

[20] Schwarz, R. J., and Friedland, B., *Linear Systems*, McGraw-Hill, NY, 1965.

[21] Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, NY, 1965.

[22] Billingsley, P., *Convergence of Probability Measures*, John Wiley, NY, 1968.

[23] Karlin, S., and Taylor, H. M., *A First Course in Stochastic Processes*, Academic Press, NY, 1975.

[24] Rabiner, L. R., Crochiere, R. E., and Allen, J. B., “Comparisons of System identification methods in the presence of high noise levels and band-limited inputs,” *Proceedings of 1978 IEEE Conference on Acoustics, Speech, and Signal Processing*, Tulsa, OK, 10-12 Apr. 1978.

[25] Halverson, D. R., “Discrete-time detection in ϕ-mixing noise,” *IEEE Trans. on Information Theory*, Vol. 26, No. 2, pp. 189-198, Mar. 1980.

[26] Bitmead, R. R., and Anderson, B. D.O., “Lyapunov techniques for the exponential stability of linear difference equations with random coefficients,” *IEEE Trans. on Automatic Control*, Vol. 25, No. 4, pp. 782-787, Aug. 1980.

[27] Pinsker, M. S., *Information and Information Stability of Random Variables and Processes*, Holden-Day, Inc., San Francisco, CA, 1964.

[28] Zarchan, P., “Representation of realistic evasive maneuvers by the use of shaping filters,” *AIAA Journal of Guidance and Control*, Vol. 2, No. 4, pp. 290-295, Jul.-Aug. 1979.

[29] Besner, E., and Shinar, J., “Output ensemble average of periodically excited linear time-varying systems,” *AIAA Journal of Guidance and Control*, Vol. 3, No. 6, pp. 587-589, Jul.-Aug. 1980.

[30] Nash, R., A., Jr., Kasper, J., F., Jr., Crawford, B. S., and Levine, S. A., “Application of Optimal Smoothing to the Testing and Evaluation of Inertial Systems and Components,” *IEEE Transactions on Automatic Control*, Vol. 16, No. 6, pp. 806-816, Dec. 1971.

[31] Pandit, S. M., and Zhang, W., "Modeling Random Gyro Drift Rate by Data Dependent Systems," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 22, No. 4, pp. 455-459, Jul. 1986.

[32] Sage, A. P., *Optimum Systems Control*, Prentice-Hall, Englewood Cliffs, NJ, 1968.

[33] James, H. M., Nichols, N. B., and Phillips, R. S. (Eds.), *Theory of Servomechanisms*, Dover, NY, 1965 edition of 1947 McGraw-Hill publication.

[34] Gradshteyn, I. S., and Ryzhik, I. M., *Table of Integrals, Series, and Products*, Academic Press, NY, 1965.

[35] Bucy, R. S., and Joseph, P. D., *Filtering for Stochastic Processes with Applications to Guidance*, Wiley-Interscience, NY, 1968.

[36] Allwright, J. C., "A lower bound for the solution of the algebraic Riccati equation of optimal control and a geometric convergence rate for the Kleinman algorithm," *IEEE Trans. on Automatic Control*, Vol. 25, No. 4, pp. 826-829, Aug. 1980.

[37] Kleinman, D. L., "On an iterative technique for Riccati equation computation," *IEEE Trans. on Automatic Control*, Vol. 13, No. 1, pp. 114-115, 1968.

[38] Kleinman, D. L., "Stabilizing a discrete, constant, linear system with applications to iterative methods for solving the Riccati equation," *IEEE Trans. on Automatic Control*, Vol. 19, No. 3, pp. 252-254, Jun. 1974.

[39] Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, NY, 1972.

[40] Barrand, A. Y., "Comments on 'the numerical solution of $A^TQ + QA = -C$'," *IEEE Trans. on Automatic Control*, Vol. 24, No. 4, pp. 671-672, Aug. 1977.

[41] Astrom, K. J., *Introduction to Stochastic Control Theory*, Academic Press, NY, 1970.

[42] Kerr, T. H., "Testing Matrices for Positive Definiteness and Semidefiniteness and Application Examples that Spawn the Need," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 10, No. 2, pp. 571-572, Mar.-Apr. 1987 (also see [69], [70]).

[43] Churchill, R. V., *Operational Mathematics*, Second Edition, McGraw-Hill Book Co., Inc., NY, 1958.

[44] Rudin, W., *Real and Complex Analysis*, McGraw-Hill Book Co., Inc., NY, 1966.

[45] Laning, J. H., and Battin, R. H., *Random Processes in Automatic Control*, Robert E. Krieger Publishing Co., NY, 1977 update and reprint of 1956 McGraw-Hill Book.

[46] Solodovnikov, V. V., *Introduction to the Statistical Dynamics of Automatic Control Systems*, Russian Version in 1952), Translated by J. B. Thomas and L. A. Zadeh, Dover, NY, 1960.

[47] Nuttall, A. H., *Multivariate Linear Predictive Spectral Analysis by Weighted Forward and Backward Averaging,* NUSC Technical Report No. AD-AO31755, 13 Oct. 1976.

[48] Lang, S. W., and McClellan, J. H., "Multidimensional MEM Spectral Estimation," *IEEE Trans. on ASSP*, Vol. 30, No. 6, pp. 880-887, Dec. 1982.

[49] Morf, M., Vieira, A., Lee, D. T. L., and Kailath, T., "Recursive Multichannel Maximum Entropy Spectral Estimation," *IEEE Trans. on Geoscience Electronics*, Vol. 16, No. 2, Apr. 1978. (See [73].)

[50] Lim, J. S., and Malik, N. A., "A New Algorithm for Two-Dimensional Maximum Entropy Power Spectrum Estimation," *IEEE Trans. on ASSP*, Vol. 29, No. 3, Jun. 1981.

[51] Marple, S. L., and Nuttall, A. H., "Experimental Comparison of Three Multichannel Linear Predictive Spectral Estimators," *Proceedings of IEEE*, Vol. 130, No. 3, pp. 218-229, Apr. 1983.

[52] Kerr, T. H., "An Invalid Norm Appearing in Control and Estimation," *IEEE Trans. on Automatic Control*, Vol. 23, No. 1, pp. 73-74, Feb. 1978.

[53] Kerr, T. H., "Correction to 'An Invalid Norm Appearing in Control and Estimation'," *IEEE Trans. on Automatic Control*, Vol. 23, No. 6, pp. 1117-1118, Dec. 1978.

[54] Kerr, T. H., "Three Important Matrix Inequalities Currently Impacting Control and Estimation Applications," *IEEE Trans. on Automatic Control*, Vol. 23, No. 6, pp. 1110-1111, Dec. 1978.

[55] Kerr, T. H., and Chin, L., "The Theory and Techniques of Discrete-Time Decentralized Filters," in *Advances in the Techniques and Technology of the Application of Nonlinear Filters and Kalman Filters*, edited by C. T. Leondes, NATO Advisory Group for Aerospace Research and Development, AGARDograph No. 256, Noordhoff International Publishing, Lieden, 1981.

[56] Kerr, T. H., "Decentralized Filtering and Redundancy Management/Failure Detection for Multi-sensor Integrated Navigation Systems," *Proceedings of the National Technical Meeting of the Institute of Navigation (ION)*, San Diego, CA, pp. 191-208, 15-17 Jan. 1985. Update under "Decentralized Filtering and Redundancy Management for Multisensor Navigation," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 23, No. 1, Jan. 1987, pp. 83-119 (corrections: May, p. 412; Jul., p. 599).

[57] Kerr, T. H., "Examining the Controversy Over the Acceptability of SPRT and GLR Techniques and Other Loose Ends in Failure Detection," *Proceedings of the American Control Conference*, San Francisco, CA, pp. 966-977, 22-24 Jun. 1983.

[58] Kerr, T. H., "The Proper Computation of the Matrix Pseudo-Inverse and Its Impact in MVRO Filtering," *IEEE Trans. on Aerospace and Electronic Systems*, Vol. 21, No.5, pp.711-724, Sept. 1985. Update and further elaboration with intermediate computations under "Computational Techniques for the Matrix Pseudoinverse in Minimum Variance Reduced-Order Filtering and Control" in *Control and Dynamic Systems-Advances in Theory and Applications*, Vol. XXVIII: Advances in Algorithms and computational Techniques for Dynamic Control Systems, Part 1 of 3, C. T. Leondes (Ed.), Academic Press, N.Y., 1988.

[59] Kerr, T. H., "Status of CR-Like Lower bounds for Nonlinear Filtering," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 25, No. 5, pp. 590-601, Sept. 1989 (Author's reply in Vol. 26, No. 5, pp. 896-898, Sept. 1990).

[60] Pervin, W. J., *Foundations of General Topology*, Academic Press, NY, 1964.

[61] Moon, P., and Spencer, D. E., *Field Theory for Engineers*, D. Van Nostrand Co., Inc., NY, 1961.

[62] Abramowitz, M., and Stegun, I A., (Ed.), *Handbook of Mathematical Functions*, National Bureau of Standards, (Applied Math, Series 55), Washington, D.C., 1964.

[63] Alexander, S. T., and Rhee, Z. M., "Spectrum Estimation Errors Due to Finite Precision Effects in Autocorrelation Based Methods," *Proceedings of Third ASSP Workshop on Spectrum Estimation and Modeling*, Boston, MA, pp. 33-36, 17-18 Nov. 1986.

[64] Kay, S. M., and Marple, S. L., "Spectrum Analysis-A Modern Perspective," *Proceedings of IEEE*, Vol. 69, No. 11, pp. 1380-1419, Nov. 1981.

[65] Rader, C. M., and Steinhardt, A. 0., "Hyperbolic Householder Transformations," *IEEE Trans. on Acoustics, Speech, and Signal Processing*, Vol. 34, No. 6, pp. 1589-1602, Dec. 1986.

[66] Anderson, B. D. O., "A Physical Basis for Krein's Prediction Formulas," *Stochastic Processes and Their Application*, Vol. 15, No. 1, pp. 133-154, 1983.

[67] Kras, G., and Potz, H., "Limiting Conditions on the Correlation Properties of Random Signals," *IRE Trans. on Circuit Theory*, Vol. 3, No. 6, pp. 282-285, Dec. 1956.

[68] Kerr, T. H., "Emulating Random Process Target Statistics (using MSF)," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 30, No. 2, pp. 556-577, Apr. 1994.

[69] Kerr, T. H., "Fallacies in Computational Testing of Matrix Positive definiteness/Semidefiniteness," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 26, No. 2, pp. 415-421, Mar. 1990.

[70] Kerr, T. H., "On Misstatements of the Test for Positive Semidefinite Matrices," *AIAA Journal of Guidance, Control, and Dynamics*, Vol. 13, No. 3, pp. 571-572, May-Jun. 1990.

[71] Kerr, T. H., "Critique of Some Neural Network Architectures and Claims for Control and Estimation," *IEEE Transactions on Aerospace and Electronic Systems*, Vol. 34, No. 2, pp. 406-419, Apr. 1998.

[72] Kerr, T. H., "Exact Methodology for Testing Linear System Software Using Idempotent Matrices and Other Closed-Form Analytic Results," *Proceedings of SPIE, Session 4473: Tracking Small Targets*, San Diego, CA, pp. 142-168, 29 Jul.-3 Aug. 2001.

[73] Nehorai, A., Morf, M., "A Relationship Between the Levinson Algorithm and the Conjugate Directions," *IEEE Trans. on Acoustics, Speech, and Signal Processing,* Vol. 31, No. 2, pp. 506-508, Apr. 1983.