Comment on 'Precision Free-Inertial Navigation with Gravity Compensation by an Onboard Gradiometer'*

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Abstract

An apparent weakness in the arguments within the derivation of [1, Appendix] is identified using explicit numerical examples which further demonstrate that the results are of limited benefit.

Our prior experience in specifying *linear system realizations* [2], [3] alerted us to an apparent problem with the new alternative procedure offered in [1, Appendix], as a potentially more straightforward way to achieve a multi-input multi-output (MIMO) *linear system realization* from a matrix power spectrum or, equivalently, from a *given* matrix correlation function by explicitly delineating both the structure and parameter values of an underlying white noise-driven Linear Time Invariant (LTI) state variable model that provides such a vector random process as its output.

The arguments for the derivation of the matrix Lyapunov equation (that the variance/cross-covariance matrix satisfies [1]) [5, pp. 222-226] are quite familiar to many analysts but fall short in Ref. [1]'s attempt to extend them in its overly concise but appealing result, where Eq. A4 equates a function of one variable on the left hand side [2] to a function of two variables on the right hand side as an obvious impossibility.

Among the beneficial results offered in [2], [3] is a non degenerate *statistically stationary* 2-channel numerical example: [3]

$$R_{yy}(\tau) = \begin{bmatrix} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \vdots & \frac{1}{4}e^{-2|\tau|} \\ \cdots & \cdots & \cdots \\ \frac{1}{4}e^{-2|\tau|} & \vdots & \frac{1}{2}e^{-|\tau|} \end{bmatrix} \Leftrightarrow S_{yy}(s) = \begin{bmatrix} \frac{2-s^2}{(4-s^2)(1-s^2)} & \frac{1}{(4-s^2)} \\ \frac{1}{(4-s^2)} & \frac{1}{(1-s^2)} \end{bmatrix} = W^T(-s)W(s), \tag{1}$$

where these second order statistics correspond to a demonstrable closed-form solution both for the intermediate (non-unique [3]) matrix spectral factorization (MSF):

$$W^T(s) \equiv \begin{bmatrix} \frac{-s-(\sqrt{7}/2)}{(2+s)(1+s)} & \frac{-1/2}{(2+s)(1+s)} \\ \frac{-s-(\sqrt{7}/2)}{(2+s)(1+s)} & \frac{3/2}{(2+s)(1+s)} \end{bmatrix} = H(sI-F)^{-1}G \tag{2}$$

(where details of accomplishing the MSF here are provided in [2, Appendix B]) and for the resulting associated linear system realization:

$$\frac{d}{dt}x(t) = F_1 x(t) + G_1 w'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ (6-\sqrt{7})/2 & -1/2 \\ -1 & 0 \\ (6-\sqrt{7})/2 & 3/2 \end{bmatrix} w'(t), \tag{3}$$

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[1] Precise regularity conditions guranteeing that the steady-state constant symmetric positive definite matrix solution P_x can be obtained from Eq. A7 with $dP_x(t)/dt \equiv 0$ is that F have only eigenvalues with real parts strictly negative and that (F, L) be a controllable pair, where $Q = LL^T$ (and where L is a factor resulting from a Choleski decomposition of Q) [4].

[2] In using this form, at this point in the derivation in [1], the assumption of stationarity had not yet been invoked as a slight mis-step of prematurely using an assumption that is invoked later.

[3] The power spectrum matrix depicted here on the right hand side of Eq. 1 consists of elements represented in the frequency domain by the bilateral Laplace transform, which relates to the Fourier transform via the substitution $s = \jmath\omega$.

and

$$y(t) = H_1 x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(t), \tag{4}$$

which is of the form

$$\frac{d}{dt}x(t) = Fx(t) + Gw'(t) \text{ and } y(t) = Hx(t), \tag{5}$$

and the corresponding $Q \triangleq GG^T$. Eq. A1 and Eq. 5 here differ slightly in that the zero mean Gaussian white noise process $w'(t)$ in Eq. 5 has the matrix identity as its covariance intensity matrix but, otherwise, corresponds to the same second order statistics for y as would be associated with Eq. A1, where $y(t) \equiv x(t)$ for Eq. A1 (corresponding to H being the identity matrix). This asserted solution as the *linear system realization* can easily be confirmed to yield the matrix power spectrum of Eq. 1 merely by using the right hand expression of Eq. 2 and multiplying out the results with the asserted parameter values of Eqs. 3 and 4 to again yield the left hand expression in Eq. 2, which, when further multiplied out as $W^T(-s)W(s)$ yields the power spectral matrix of Eq. 1 as a cross-check.

We now attempt to apply the steps for achieving a *linear system realization* provided in [1] to the correlation function specified in Eq. 1, where it is more convenient to use the time domain representation in Eq. 1 instead. In attempting to take the derivative of the correlation function of Eq. 1, to be evaluated at $\tau = 0$ (corresponding to $\lim_{t \to t_o}$), as called for in Eq. A5, yields:

$$\lim_{\tau \to 0} \frac{d}{d\tau} R_{yy}(\tau) = \frac{d}{d\tau} \left[\begin{array}{c|c} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \frac{1}{4}e^{-2|\tau|} \\ \cdots & \cdots \quad \cdots \\ \frac{1}{4}e^{-2|\tau|} & \frac{1}{2}e^{-|\tau|} \end{array} \right] \Bigg|_{\tau=0}, \tag{6}$$

but this presents a problem since the terms constituting the elements of the matrix, as a composite function, are not differentiable at $\tau = 0$ since the absolute value of τ, being $|\tau|$, itself is not differentiable at the origin and we are stymied by being unable to proceed any further using this approach. However, to illustrate what further problems are to be encountered, suppose that an even more benign matrix correlation function were being utilized such as that corresponding to the familiar *ergodic* random process [6, Chapt. 9] with both channels being independent, as a possible further simplification, then we have:

$$x_1(t) = A\sin(\omega t + \theta), \quad x_2(t) = B\sin(\omega t + \theta), \tag{7}$$

with A, B, and ω being deterministic non-zero constants, and θ being a zero mean random variable uniformly distributed over $[-\pi, \pi]$. Its matrix correlation function would be:

$$R_{yy}(\tau) = \begin{bmatrix} A\cos(\omega\tau) & 0 \\ 0 & B\cos(\omega\tau) \end{bmatrix}, \tag{8}$$

and, upon attempting to differentiate it with respect to τ at $\tau = 0$, this step can now be accomplished as:

$$\lim_{tau \to 0} \frac{d}{d\tau} R_{yy}(\tau) = \begin{bmatrix} -A\omega\sin(\omega\tau) & 0 \\ 0 & -B\omega\sin(\omega\tau) \end{bmatrix} \Bigg|_{\tau=0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{9}$$

and we are one step further but stuck again and unable to take the approach of [1] to fuition. These two numerical examples are extremely well behaved and exhibit all the properties of a valid correlation function matrix [3, Sec. IIB]; however, they reveal the weaknesses of this asserted new approach although the initial covariances can be found for the above two examples, respectively, as:

$$\lim_{\tau \to 0} R_{yy}(\tau) = \lim_{\tau \to 0} \left[\begin{array}{c|c} \frac{1}{6}e^{-2|\tau|} + \frac{1}{6}e^{-|\tau|} & \frac{1}{4}e^{-2|\tau|} \\ \cdots & \cdots \quad \cdots \\ \frac{1}{4}e^{-2|\tau|} & \frac{1}{2}e^{-|\tau|} \end{array} \right] = \left[\begin{array}{c|c} \frac{1}{3} & \frac{1}{4} \\ \cdots & \cdots \quad \cdots \\ \frac{1}{4} & \frac{1}{2} \end{array} \right], \tag{10}$$

and as:

$$\lim_{\tau \to 0} R_{yy}(\tau) = \lim_{\tau \to 0} \begin{bmatrix} A\cos(\omega\tau) & 0 \\ 0 & B\cos(\omega\tau) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}. \tag{11}$$

Since the means of both examples are zero vectors, here we have $R_{yy}(0) \stackrel{\triangle}{=} E[x(t_o)x^T(t_o)] = P_x(t_o)$ and in both examples the resulting $P_x(t_o)$ is non-singular and invertable.

Looking further at Eq. A5 for what is needed to continue with the above two examples to specify a corresponding F using the newly posed methodology, even with the matrix $P_x(t_o)$ specified and nonsingular in both cases, one cannot proceed in specifying F from Eq. A5 and consequently Eq. A8 is useless as well. However, if one were attempting to solve Eq. A7 for the necessary constant $P_x(t_o)$ using the standard prior approach, one first needs to know F and Q. While it is true that for a Kalman filtering situation, where any symmetric positive definite matrix suffices for the initial condition covariance to start integrating out the Riccati equation, which exponentially asymptotically converges to the correct solution under fairly mild regularity conditions [7] even if the exact initial covariance matrix is unknown, an arbitrary symmetric positive definite $P_x(t_o)$ hypothesized starting value does not necessarily correspond to the steady state solution of Eq. A7 (and the rate of convergence in just integrating it out with time is much slower than that of the somewhat similar looking Riccati equation) to obtain the necessary final result for $P_x(t_o)$ that is needed to represent the corresponding stationary random process. Hence, one cannot use Eq. A5 to determine F and eventually Q from Eq. A8 for even these two benign examples, which points to a rather severe limitation in applicability of the new approach offered in [1, Appendix].

References

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