Apparent Drawbacks that Exist in the Underlying Structure of Covariance Intersection

Thomas H. Kerr III, Ph.D.
TeK Associates, P.O. Box 459, 9 Meriam St., Suite 7-R, Lexington, MA 02420-5312
e-mail: thomas_h_kerr@msn.com.

Abstract

A counterexample is presented to a result claimed in a proof in [1], pertaining to using this new approach to Covariance Intersection (CI). Other researchers have already demonstrated certain problems that exist with earlier versions of CI, as summarized in a survey [7] of the previous CI approaches encountered in *Target tracking* applications. We alert readers to investigations along similar lines from the field of navigation that were apparently overlooked in [7] that convey different results for similar problems of ascertaining ellipsoidal overlap and for combining the estimates from two or more Kalman filters, each representing a different sensor's output that has already been processed locally before combining the intermediate information afterwards for a better final global estimate.

1.Introduction and Overview of the New Approach to CI

This technical note offers a counterexample to the use of the results of [1] in this new Covariance Intersection (CI) approach. An expression for the estimate that results from combining two prior (assumed) independent estimates consisting of ($\hat{\mathbf{x}}_1$, $\mathbf{P}_{\mathbf{aa}}$) and ($\hat{\mathbf{x}}_2$, $\mathbf{P}_{\mathbf{bb}}$) is of the following well-known form ([19], and as summarized from Eq. 2 to the end of Sec. II of [1]):

$$\hat{\mathbf{x}}_c = \mathbf{K}_1\hat{\mathbf{x}}_1 + \mathbf{K}_2\hat{\mathbf{x}}_2 = \mathbf{P}_{\mathbf{bb}}\mathbf{P}_{\mathbf{cc}}^{-1}\hat{\mathbf{x}}_1 + \mathbf{P}_{\mathbf{aa}}\mathbf{P}_{\mathbf{cc}}^{-1}\hat{\mathbf{x}}_2 = \mathbf{P}_{\mathbf{aa}}^{-1}\mathbf{P}_{\mathbf{cc}}\hat{\mathbf{x}}_1 + \mathbf{P}_{\mathbf{bb}}^{-1}\mathbf{P}_{\mathbf{cc}}\hat{\mathbf{x}}_2 . \quad (1)$$

The exact covariance corresponding to the above, with the *assumption of unbiased* estimates, is:

$$\tilde{\mathbf{P}}_{\mathbf{cc}} \triangleq E\left[\tilde{\mathbf{x}}_c\tilde{\mathbf{x}}_c^{\mathbf{T}}\right]. \quad (2)$$

The above expression of Eq. 1, consisting of the indicated weighted combination of the two prior linear estimates and utilizing the accompanying covariances $\mathbf{P}_{\mathbf{aa}}$ and $\mathbf{P}_{\mathbf{bb}}$, seeks to use an acceptable approximate covariance $\mathbf{P}_{\mathbf{cc}}$ that conservatively suffices in its role of making Eq. 1 be a useful single combined estimator if and only if $\mathbf{P}_{\mathbf{cc}}$ is a *consistent covariance* (in the matrix positive semi-definite sense) by satisfying the following required upper bound criterion ([1, Eq. 4]):

$$\mathbf{P}_{\mathbf{cc}} \geq \tilde{\mathbf{P}}_{\mathbf{cc}} , \quad (3)$$

and the quest for a satisfactory *consistent covariance* upper bound motivated use of this particular expression:

$$\mathbf{P}_{\mathbf{cc}}(\omega) = \left[\omega\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}, \quad (4)$$

(advocated for use in Eqs. 9 and 10 of [1]), which when substituted back into Eq. 1 yields:

$$\hat{\mathbf{x}}_c(\omega) = \mathbf{K}_1(\omega)\hat{\mathbf{x}}_1 + \mathbf{K}_2(\omega)\hat{\mathbf{x}}_2 = \omega\left[\omega\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}\mathbf{P}_{\mathbf{aa}}\hat{\mathbf{x}}_1 + (1-\omega)\left[\omega\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}\mathbf{P}_{\mathbf{bb}}\hat{\mathbf{x}}_2. \quad (5)$$

Ref. [1] then advocates optimizing ω^* in the above to minimize trace of Eq. 4 (cf., [1, Eq. 14]):

$$\mathrm{Trace}\left\{\mathbf{P}_{\mathbf{cc}}(\omega)\right\} = \mathrm{Trace}\left\{\left[\omega\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}\right\}, \quad (6)$$

and goes further to provide Theorem 2 [1, p. 1881] that claims the global minimum occurs for:

$$\omega^* \in [0,1]. \quad (7)$$

The resulting optimized ω^* is then substituted back, respectively, into the expressions of Eqs. 4 and 5 (even when the constituent component estimates are no longer independent and the cross-covariance $\mathbf{P}_{\mathbf{ab}}$ may be unknown or inaccessible) to be the **best** fused estimator of the form:

$$\hat{\mathbf{x}}_c^* = \mathbf{K}_1^*\hat{\mathbf{x}}_1 + \mathbf{K}_2^*\hat{\mathbf{x}}_2 = \omega^*\left[\omega^*\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega^*)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}\mathbf{P}_{\mathbf{aa}}\hat{\mathbf{x}}_1 + (1-\omega^*)\left[\omega^*\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega^*)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}\mathbf{P}_{\mathbf{bb}}\hat{\mathbf{x}}_2 , \quad (8)$$

with the corresponding accompanying associated covariance:

$$\mathbf{P}_{\mathbf{cc}}^* \equiv \mathbf{P}_{\mathbf{cc}}(\omega^*) = \left[\omega^*\mathbf{P}_{\mathbf{aa}}^{-1} + (1-\omega^*)\mathbf{P}_{\mathbf{bb}}^{-1}\right]^{-1}. \quad (9)$$

Since the original two estimates and accompanying covariances are all real quantities, clearly, the two expressions of Eqs. 8 and 9 also need to yield exclusively real results also. If one obtained a complex answer for ω^* as the solution that globally minimizes the criterion of Eq. 6, this would constitute a counterexample to what is claimed (last sentence 2 paragraphs before Sec. II in [1]) and ostensibly proved in Theorem 2 [1, p. 1881], namely, that the optimizing ω^* either lies on the two boundary points 0 or 1 or lies within the interior of [0, 1]. Once this was *definitively established* according to [1], they could then turn their attention in [1] to just searching over the compact interval [0, 1] for the minimum that is guaranteed to exist from first principles of *real analysis* for this continuous cost criterion of Eq. 6 (as the composite of the trace and the matrix inverse) since scalar continuous functions always achieve both a maximum and a minimum on a compact set. However, only Theorem 2 of [1] asserts that such a minimum is also the global minimum (otherwise it wouldn't be of interest since this criterion of using the trace of the associated covariance was specifically chosen within [1] to be consistent with what was already correspondingly used in the derivation of the underlying Kalman filters from which the prior constituents ($\mathbf{x_1}$, $\mathbf{P_{aa}}$) and ($\mathbf{x_2}$, $\mathbf{P_{bb}}$) were obtained). If this local minimum were indeed also the global minimum, we would have no further objections here. However, Example 1 below serves as a counterexample to the global optimization assertion [1, Thm. 2] since it yields a complex answer for ω^*. This is our main point here.

Similarities and connections to other tests for ellipsoid overlap and pre-existing warnings regarding other earlier Covariance Intersection approaches are discussed in Sec. 4.

2.A Numerical Counterexample

A closed-form evaluation will now be provided for this new version of CI [1] for the simple numerical example below that exposes a difficulty with using this CI approach that has not been previously publicized.

<u>Example 1</u>:

$$P_{aa} = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix};\ P_{bb} = \begin{bmatrix} 3 & 1.5 \\ 1.5 & 9 \end{bmatrix} \text{ and } P_{bb} - P_{aa} = \begin{bmatrix} 1 & 1 \\ 1 & 7 \end{bmatrix} > 0, \text{ where } \left[P_{bb} - P_{aa}\right] \text{ has } \lambda = 2, 6. \tag{10}$$

Notice that both P_{aa} and P_{bb} above are positive definite, as with all non-degenerate covariances.

To explicitly demonstrate this new CI technique, we seek to apply the solution of Problem 2 (of [1]) to Eq. 10 above yielding the following abbreviated intermediate steps as we seek to directly solve for the optimizing ω^* that should minimize the trace of P_{cc} below (according to the procedure of [1, Eq. 16] using the derivative convention stated in the footnote on the next page):

$$P_{aa}^{-1} = \begin{bmatrix} \frac{8}{15} & \frac{-2}{15} \\ \frac{-2}{15} & \frac{8}{15} \end{bmatrix};\ P_{bb}^{-1} = \begin{bmatrix} \frac{12}{33} & \frac{-2}{33} \\ \frac{-2}{33} & \frac{4}{33} \end{bmatrix}; \tag{11}$$

$$P_{cc} = \begin{bmatrix} \frac{8\omega}{15} + \frac{12(1-\omega)}{33} & \frac{-2\omega}{15} - \frac{2(1-\omega)}{33} \\ \frac{-2\omega}{15} - \frac{2(1-\omega)}{33} & \frac{8\omega}{15} + \frac{4(1-\omega)}{33} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{84\omega+180}{495} & \frac{-36\omega-30}{495} \\ \frac{-36\omega-30}{495} & \frac{204\omega+60}{495} \end{bmatrix}^{-1} = \frac{165}{(28\omega+60)(68\omega+20)-(12\omega+10)^2} \begin{bmatrix} (68\omega+20) & (12\omega+10) \\ (12\omega+10) & (28\omega+60) \end{bmatrix}; \tag{12}$$

$$\mathrm{tr}\left[P_{cc}\right] = \frac{165\left[(68\omega+20)+(28\omega+60)\right]}{(28\omega+60)(68\omega+20)-(12\omega+10)^2} = \frac{165\cdot 4(24\omega+20)}{(28\omega+60)(68\omega+20)-(12\omega+10)^2}, \tag{13}$$

with critical points obtained from setting $\frac{\partial}{\partial\omega}\mathrm{tr}\left[P_{cc}\right] = 0$ and solving for the zeros of:

$$0=(24\omega+20)\left[(28)(68\omega+20)+(28\omega+60)(68)\text{-}24(12\omega+10)\right]\text{-}24\left[(28\omega+60)(68\omega+20)\text{-}(12\omega+10)^2\right]$$

$$=(24\omega+20)(3{,}520\omega+4{,}400)\text{-}24\left[1{,}760\omega^2+4{,}400\omega+1{,}100\right]=61{,}600+70{,}400\omega+42{,}240\omega^2, \quad (14)$$

a quadratic equation; with solutions being:

$$\omega^*=\frac{-b\pm\sqrt{b^2\text{-}4ac}}{2a}=\frac{\text{-}70{,}400\pm\sqrt{(70{,}400)^2\text{-}4(42{,}240)(61{,}600)}}{2(42{,}240)}=\frac{\text{-}70{,}400\pm\sqrt{\text{-}5{,}451{,}776{,}000}}{2(42{,}240)}. \quad (15)$$

The above Eq. 15 possesses no *solutions*† *over the real field* and, in particular, has no solution within the predicted interval [0,1] and so the new CI approach of [1] is apparently stymied here and can proceed no further for this numerical example corresponding to $\mathbf{P_{bb}}$-$\mathbf{P_{aa}}$ being strictly positive definite. We did not anticipate that this new CI approach of [1] would have such problems when the containment condition demonstrated in Eq. 10 (i.e., $\mathbf{P_{bb}}$-$\mathbf{P_{aa}}$>**0** or, equivalently, $\mathbf{P_{bb}}$>$\mathbf{P_{aa}}$) was strictly met (a condition that was present but down played in the proof of [1]); so we were surprised when it failed to yield an adequately real solution for ω^*.

While it is indeed true that a continuous function of ω (such as the matrix inverse, constituting the RHS of Eq. 4, composed with the trace operation of Eq. 6) over a compact interval like [0,1] achieves its minimum there, we reject the suggesting that we merely confine optimization to be over [0,1] since such a constraint would, in general, only yield a local minimum. The proofs of [1] supposedly guarantee that by merely optimizing the expression of the LHS of Eq. 3 over just the interval [0, 1], this should also be the global minimum. However, this example 1 demonstrates this claim of [1] to be false. If we can not even get correct answers from the technique of [1] when we are looking closely at this non-pathological situation, how can we trust it to do the right thing in the more general case when we are not looking at it but merely seeking to apply the associated constrained optimization code in an automated fashion? As seen for the typical examples considered here, such blind faith reliance would be dangerous (because it didn't work).

According to [1], only after minimizing the above Eq. 13 can the two optimal gains and resulting associated optimal covariance $\mathbf{P_{cc}}$ be explicitly evaluated (by substituting the result of Eq. 15 back into Eqs. 8 and 9, respectively, where Eq. 9 has already been simplified to be Eq. 12) using CI. The above numerical example exhibits a result that is therefore inconsistent with what the CI approach of [1] asserts (contrary to what is expected as supposedly proved in [1, Thm. 2]) so [1] appears to *not* work as it should in all cases. A natural question is why didn't it work.

By insights availed from [3, p. 1141], it is recognized that for two ellipsoids sharing a common center, the differences in covariances (such as those depicted in Eq. 10) serve as a test for full containment of one ellipsoid within another if and only if the matrix difference between two covariance matrices is positive definite. Numerical tests for positive definiteness/semi-definiteness are well known [4] and can serve as a warning of this same condition where the approach of [1] will likely fail, as depicted here in the numerical example above.

While it can be argued that, initially, there is no apparent physical reason why these initial covariance matrices should exhibit any partial ordering between them. Two synchronized decentralized estimates of the same target state vector, as viewed from different sensors with, perhaps, different perspective views, different segments of the electromagnetic spectrum utilized (to exploit inherent target characteristics), and different noise contamination intensities is but one example of why unaltered initial covariances would not necessarily exhibit such a partial-order-

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† We formed $\frac{d}{d\omega}\left(\frac{u}{v}\right)=\frac{v\frac{du}{d\omega}-u\frac{dv}{d\omega}}{v^2}$ and set $v\frac{du}{d\omega}-u\frac{dv}{d\omega}=0\Leftrightarrow u\frac{dv}{d\omega}-v\frac{du}{d\omega}=0$, and so Eq. 5 here was effectively multiplied throughout by –1 but that doesn't alter the location of the roots of the resulting quadratic equation.

ing in a completely general sensor fusion application but, instead, likely be skewed off from each other in tilt and overall size. However, use of collocated conventional radar along with laser radar may yield one target ellipsoid contained entirely within another due to the greater resolution and smaller azimuth error incurred for laser optics.

3.Offering An Alternative CI Interpretation based on a Different Matrix Inequality

A simpler approach is now explored here, based on convexity of the matrix inverse over positive definite matrices [2], as:

$$\left[\omega A+(1\text{-}\omega)B\right]^{-1}\le \omega A^{-1}+(1\text{-}\omega)B^{-1}\ \text{ for all }\ 0\le\omega\le 1. \tag{16}$$

When this result is applied to the expression of Eq. 4 above in seeking a covariance upper bound as in Eq. 3, the following results, requiring no matrix inversions at all for the RHS vs. a LHS (from [1, Eq. 4]) that does:

$$P_{cc}(\omega)\ \triangleq\ \left[\omega P_{aa}^{-1}+(1\text{-}\omega)P_{bb}^{-1}\right]^{-1}\le\ \omega P_{aa}+(1\text{-}\omega)P_{bb}\triangleq P'_{cc}(\omega)\ \text{ for all }\ 0\le\omega\le 1. \tag{17}$$

Notice that , $P'_{cc}(\omega)$ on the RHS represents an upper bound that is easier and more convenient to obtain and, moreover, by performing trace operations throughout Eq. 17, also yields a corresponding simple upper bound on the trace of P_{cc} as:

$$\text{tr}\left[P_{cc}\right]\ =\ \text{tr}\left[\omega P_{aa}^{-1}+(1\text{-}\omega)P_{bb}^{-1}\right]^{-1}\le\ \omega\ \text{tr}\left[P_{aa}\right]+(1\text{-}\omega)\ \text{tr}\left[P_{bb}\right]=\text{tr}\left[P'_{cc}(\omega)\right]\ \text{ for all }\ 0\le\omega\le 1. \tag{18}$$

However, although it is rigorous, this path is not a panacea since the resulting bound is likely to be slightly coarser (i.e., larger), in general than what would be provided by the optimizing CI approach of [1] (when the approach of [1] works). The benefit of this alternate approach is that (1) it requires no matrix inversions at all in its numerical evaluation and (2) it is always true for all $0\le\omega\le 1$ without any qualifications. The convexity property itself delineates the interval of primary interest to be [0, 1] and not because of some auxiliary theorem, as with the approach of [1]. To complete the demonstration of this alternate procedure as providing a closed-form answer for this same example for the numerical parameters of **Ex.** 1, the associated bound using the RHS of Eq. 18 is:

$$P'_{cc}(\omega)\triangleq\ \omega P_{aa}+(1\text{-}\omega)P_{bb}=\omega\begin{bmatrix}2 & 0.5\\ 0.5 & 2\end{bmatrix}+(1\text{-}\omega)\begin{bmatrix}3 & 1.5\\ 1.5 & 9\end{bmatrix}=\begin{bmatrix}3\text{-}\omega & 1.5\text{-}\omega\\ 1.5\text{-}\omega & 9\text{-}7\omega\end{bmatrix}; \tag{19}$$

(and the RHS of Eq. 19 is again a *consistent covariance* since it is an upper bound of the actual covariance already bounded above by P_{cc} in Eq. 3) and furthermore

$$\text{Tr}\left[P'_{cc}(\omega)\right]=\omega\ \text{Tr}\left[P_{aa}\right]+(1\text{-}\omega)\ \text{Tr}\left[P_{bb}\right]=\omega\ (4)+(1\text{-}\omega)(12)=12\text{-}8\omega. \tag{20}$$

For Ex. 1, the more easily obtained bounds of both Eqs. 19 and its corresponding 20 are available and the bound of Eq. 20 is minimized over $0\le\omega\le 1$ for $\omega=1.00$, as verified by inspection here (and the same MatLab® computer code already developed for the CI of [1], as conveyed by the authors of [1] in a private correspondence pertaining to this technical note, can be used as a computational algorithm) yielding:

$$P'_{cc}(1.00)\ \triangleq\begin{bmatrix}2 & 0.5\\ 0.5 & 2\end{bmatrix}. \tag{21}$$

The appropriate gains associated with this more conservative covariance bound of Eq. 21 can now be completely specified as in Eq. 8 but with the result of Eq. 21 replacing that of Eq. 9 as:

$$K_1(\omega)\triangleq\omega\ P'_{cc}P_{aa}^{-1}=\ (1.00)\begin{bmatrix}2 & 0.5\\ 0.5 & 2\end{bmatrix}\begin{bmatrix}\frac{8}{15} & \frac{-2}{15}\\ \frac{-2}{15} & \frac{8}{15}\end{bmatrix}=\left(\frac{1}{15}\right)\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}\begin{bmatrix}4 & -1\\ -1 & 4\end{bmatrix}=\left(\frac{1}{15}\right)\begin{bmatrix}15 & 0\\ 0 & 15\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{22}$$

$$K_2 \triangleq (1\text{-}\omega) P'_{cc} P_{bb}^{-1} = (0.00)\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix}\begin{bmatrix} \frac{12}{33} & \frac{-2}{33} \\ \frac{-2}{33} & \frac{4}{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \quad (23)$$

The CI optimal weightings of [1] for **Ex.** 1 would use the solution of Eq. 15 in Eqs. 22, 23 in place of 1.00 and in place of the matrix of Eq. 19, would use the expression of Eq. 12 with the same value of Eq. 19 inserted throughout if it were in fact between 0 and 1 (which it is not). The procedure of [1] cannot be applied for this example since [1, Thm. 2] is evidently violated.

I offer a scalar example that should allow a reader to visualize the problem with CI more easily. For a scalar case situation, the ideal formula for the associated covariance of two fused estimates when the two underlying constituent estimates are independent looks like the formula for combining two resistances in parallel (and is known to be less than the smaller of the two). This result is intuitively appealing and consistent with the tenets of Kalman Filtering. The following two algorithms: (1) the CI covariance of [1] using any omega, and (2) the alternative expression for the covariance, offered by me, using convexity of the matrix inverse over positive definite matrices both yield a covariance that is larger than the smallest of the two original covariances unless ω is either 0 or 1, in which case it has a resulting covariance that is identical to the smallest one.

I.For two ideal independent estimates, the resulting covariance for the combined estimates would be:

$$\tilde{p}_{cc} = \frac{1}{\frac{1}{p_1}+\frac{1}{p_2}} = \frac{1}{\frac{p_2}{p_1 p_2}+\frac{p_1}{p_1 p_2}} = \frac{1}{\frac{(p_2+p_1)}{p_1 p_2}} = \frac{p_1 p_2}{(p_2+p_1)} = \left\{ \begin{array}{l} \frac{p_2}{\left(\frac{p_2}{p_1}+1\right)} \le p_2, \text{ for } p_1 > 0,\ p_2 > 0 \\ \frac{p_1}{\left(1+\frac{p_1}{p_2}\right)} \le p_1, \text{ for } p_1 > 0,\ p_2 > 0 \end{array} \right\} \le \min\{p_1, p_2\}$$

II.For the new CI algorithm of [1] for $0 < \omega < 1$, the covariance for the fused estimates would be:

$$p_{cc} = \frac{1}{\frac{\omega}{p_1}+\frac{(1\text{-}\omega)}{p_2}} = \frac{1}{\frac{(\omega p_2 + (1\text{-}\omega) p_1)}{p_1 p_2}} = \left\{ \begin{array}{l} \frac{p_1 p_2}{(\omega p_2 + (1\text{-}\omega) p_1)} \le \frac{p_1 p_2}{(\omega p_2 + (1\text{-}\omega) p_2)} = p_1 \text{ when } p_2 \le p_1 \\ \frac{p_1 p_2}{(\omega p_2 + (1\text{-}\omega) p_1)} \le \frac{p_1 p_2}{(\omega p_1 + (1\text{-}\omega) p_1)} = p_2 \text{ when } p_1 \le p_2 \end{array} \right\} = \max\{p_1, p_2\}$$

III.For the CI alternative path based on convexity of the matrix inverse over positive definite matrices for $0 < \omega < 1$, the covariance for the fused estimates would be:

$$p_{cc} = \frac{1}{\frac{\omega}{p_1}+\frac{(1\text{-}\omega)}{p_2}} \le \omega p_1 + (1\text{-}\omega) p_2 \triangleq p'_{cc} = \left\{ \begin{array}{l} \omega p_1 + (1\text{-}\omega) p_2 \le \omega p_2 + (1\text{-}\omega) p_2 = p_2 \text{ when } p_1 \le p_2 \\ \omega p_1 + (1\text{-}\omega) p_2 \le \omega p_1 + (1\text{-}\omega) p_1 = p_1 \text{ when } p_2 \le p_1 \end{array} \right\} = \max\{p_1, p_2\}$$

So both these two CI covariance calculation paths yield the same answer for the scalar case!

Lemma used in III (if convexity of the inverse were not already granted as true):

$$\frac{p_1 p_2}{[\omega p_2 + (1\text{-}\omega) p_1]} \le \omega p_1 + (1\text{-}\omega) p_2$$

Proof: (Retrace the following steps backwards.)

$$p_1 p_2 \le [\omega p_1 + (1\text{-}\omega) p_2][\omega p_2 + (1\text{-}\omega) p_1] = \omega^2 p_1 p_2 + \omega(1\text{-}\omega) p_2^2 + \omega(1\text{-}\omega) p_1^2 + (1\text{-}\omega)^2 p_1 p_2$$

$$0 \le -p_1 p_2 + \omega^2 p_1 p_2 + \omega(1\text{-}\omega)\, p_2^2 + \omega(1\text{-}\omega)\, p_1^2 + (1\text{-}\omega)^2\, p_1 p_2 = -2\omega(1\text{-}\omega)\, p_1 p_2 + \omega(1\text{-}\omega)\, p_2^2 + \omega(1\text{-}\omega)\, p_1^2$$

$$0 \le -2 p_1 p_2 + p_2^2 + p_1^2 = \left[p_2\text{-}p_1\right]^2$$

since $0 < \omega < 1 \Rightarrow \omega(1\text{-}\omega) > 0$.

From the above II and III with similar manipulations, it is also easy to show that:

$$\min\{p_1, p_2\} \le p_{cc} \le \max\{p_1, p_2\}$$

and, likewise, that:

$$\min\{p_1, p_2\} \le p'_{cc} \le \max\{p_1, p_2\}.$$

Compare the above to

This is not a very satisfying outcome of applying either of these two CI strategies.

4.Conclusions, Summary Perspectives, and Beyond

Uncertainty being summarized as covariance ellipsoids normally only rigorously arises for the case of standard linear systems with Gaussian initial conditions independent of the additive Gaussian process and measurement noises (with known covariance intensities) and outfitted with a pure Kalman filter as an optimal linear estimator, mechanized either in a decentralized or centralized manner. Ellipsoidal confidence regions of constant pdf would also reasonably represent the class of *elliptical distributions*‡ and the conditional and marginal distributions of the *exponential family* of distributions but they typically do not arise (yet) in the standard estimation and filtering context of *most* normal target tracking or navigation applications.

Caution is conveyed here regarding the result of [1] apparently not applying when one ellipsoid is wholly contained within the other. An apparent hole in the applicability of the Covariance Intersection (CI) approach of [1] was illustrated here using an explicit numerical example. A novelty is that the two participating covariances being related as

$$\mathrm{P}_1 < \mathrm{P}_2 \tag{24}$$

was historically encountered by this author in [5] before being able to specify a test for ellipsoid overlap (in n-dimensions) when the centers of the respective ellipsoids differ, where the particular covariance matrix, P_1, in this case the solution of the Riccati equation is so related to the other covariance matrix, P_2, in this case the solution of the Lyapunov equation. Remarkably, the result of [5] parallels (but is not identical to) what is done in Chen et al [1]. However, the proof of Eq. 22 was easily accomplished in Lemma 5.1 of [5] by just taking the synchronous difference of the two respective matrix differential equations that describe their evolution in time (in either continuous- or discrete-time) by demonstrating that the difference is always positive definite (as it evolves for all time steps k>0) as the positive definite matrices within the bracket below, as pre- and post-multiplied by a non-singular matrix and its transpose (yielding a positive semi-definite intermediary matrix) and added to a positive definite matrix yielding a positive definite matrix result as:

$$\left[\mathrm{P}_2(k+1) - \mathrm{P}_1(k+1\,|\,k)\right] = \tag{25}$$

$$\Phi(k+1,k)\left[\mathrm{P}_2(k) - \mathrm{P}_1(k\,|\,k)\right]\Phi^{\mathrm{T}}(k+1,k) + \Phi(k+1,k)\mathrm{P}_1(k\,|\,k-1)\mathrm{H}^{\mathrm{T}}\left[\mathrm{H}\,\mathrm{P}_1(k\,|\,k-1)\mathrm{H}^{\mathrm{T}} + \mathrm{R}(k)\right]^{-1}\mathrm{H}\,\mathrm{P}_1(k\,|\,k-1)\Phi^{\mathrm{T}}(k+1,k)$$

The associated optimization problem in [5] has great similarity to that in [1] since the associated Lagrange multiplier was also merely a scalar. In the case of posing the simpler problem of a one dimensional test for the overlap of scalar Gaussian confidence intervals in [6] to show how the same test then generalizes to n-dimensions, as a test for the overlap of Gaussian Ellipsoidal Confidence Regions, the version of the test in [6] (that was simpler than that in [5]) reveals other as-

‡ *Elliptical distributions* have recently been used in attempting to compensate for the ground clutter seen by airborne radar.

pects that are similar in form to the structure encountered in [1] in enabling a closed-form answer to the optimization that also proceeds in both [5] and [6], after just optimizing the selection of λ^* along a scalar direction (i.e., the essence of the main result of [1]). When the containment condition is strictly satisfied, the numerical example of Sec. 2 failed to satisfy the expected condition on the optimum value of ω^* that it fall somewhere within the real interval [0,1]. (Ref. [1] also lacks any corresponding numerical description or, alternatively, any explicit reference for the illustrative planar examples presented in Figs. 1 and 2 of [1], respectively, of the intersection *inscribing* and *circumscribing* ellipses that supposedly motivates how their approach should behave.) The proofs in [1], although beautifully constructed, may perhaps be a mere tautology. An alternative approach offered in Sec. 3 guarantees that the resulting fused estimate and associated CI covariance are exclusively real without exception. We still caution against CI.

Ref. [7] is an excellent overview, in depth numerical ranking, and clear interpretation of all of the various approaches to *sensor fusion* that have occurred *in the target tracking literature* over the last two decades and culminates in an algorithmic improvement to CI that [7] attributes to the pioneering work of the late Fred C. Schweppe's *unknown but bounded* approach [8], where, instead of embracing the assumption of Gaussian noises being present, uses an *entirely deterministic approach* of circumscribing the set of potential outcomes arising at each discrete-time step of a linear system's output within an ellipsoid. This approach, although creative, was notoriously conservative and was never used in applications nor was it admired as frequently as what is reported on in [9], as more representative of Schweppe's genius. (Overlooked in [7], parallel developments were occurring in *navigation*, as similar techniques were being examined, e.g., [10]-[13]; criticized by Larry Levy JHU/APL [14]; and refuted [15] for Jason Spyer's **exact** 1979 version of *decentralized estimation* (also see [16]). Ref. [7] demonstrates that the approximate algorithms of the CI approach should only be used with extreme caution since *uncertainty increases as more measurements are used* (and fused) as a counterintuitive, extremely unsettling result stated and proved simply and convincingly in [7, following Eq. 22].

Please see [18] for an elegant solution to *assessing whether two n-dimensional ellipsoids overlap* that is easy to understand and even straightforward to test for numerically (using only positive definiteness\semi-definiteness tests along with eigenvalue-eigenvector calculation), obtained by exploiting features of an ellipsoid representation familiar in computer graphics applications [17, pp. 479-481] for the two ellipsoids of interest (but less familiar to most engineers) as, respectively:

$$\mathbf{x}^{\mathbf{T}}\overbrace{\mathbf{M}\mathbf{S}_1\mathbf{M}^{\mathbf{T}}}^{\mathbf{A}}\mathbf{x}=0,\ \text{and } \mathbf{S}_1 \triangleq \begin{bmatrix} (1/2)P_1^{-1} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \text{ with offset: } \mathbf{M}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\bar{x}_1 & -\bar{x}_2 & -\bar{x}_3 & 1 \end{bmatrix} \text{ where } \mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix};$$

$$\mathbf{x}^{\mathbf{T}}\overbrace{\mathbf{S}_2}^{\mathbf{B}}\mathbf{x}=0,\ \text{and } \mathbf{S}_2 \triangleq \begin{bmatrix} (1/2)P_2^{-1} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix} \text{ with no offset (without any loss of generality);}$$

then must solve for λ in $\mathbf{x}^{\mathbf{T}}\mathbf{A}\left[\lambda\mathbf{I}_{4\times 4}-\mathbf{A}^{-1}\mathbf{B}\right]\mathbf{x}=0$ (obtained by combining both equations above) to determine whether or not the underlying two 3-D ellipsoids of primary interest above do overlap. Corresponding compatible eigenvectors also need to be found and tested for consistency to complete the test of [18]. The clear result of [18] was obtained by embedding a test for the overlap of n-dimensional ellipsoids into a test that is performed in an associated (n+1)-dimensional space (which, coincidently, the analysis of [5] and [6] also did). However, the resulting test in [18] appears to be simpler to implement as a lesser computational burden (than that of [5] obtained 30 years earlier) **except that** [18] apparently overlooks the *intermediate iterative techni-*

ques also needed internally within the software package used to solve for the necessary candidate eigenvalues and eigenvectors used in making the determination (and additional logic still needs to be programmed for scaling the last component of **x** to be 1 [consistent with the methodology of embedding the n-dimensional problem into (n+1)-dimensions] and for other aspects of unwinding or interpreting a final decision regarding presence or absence of overlap). Ref. 18, not needing any condition of Eq. 22 to be satisfied, is for a more general case than treated in [5], [6]; however, the numerical calculations of [5], [6] are tailored for a standalone real-time decision.

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