Department of Electrical Engineering

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Ph.D. Thesis Proposal

Analysis sf the Control of Nonlinear Systems subjected to Stochastic Disturbances

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Table of Contents

I. Introduction

Muck of the phenomena of the real world is nonlinear and should be analyzed using nonlinear differential equations without recourse to linearized versions of these differential equations which, when solved, yield answers strikingly different from the observed phenomena and, hence, having little value as a mathematical model. Many of the control systems encountered today have essential nonlinearities resulting from the physical limitations of the devices employed. In order to obtain meaningful results from the mathematical model, satisfactory methods of analysis must be found for both the case of deterministic inputs and the case of stochastic inputs. Suitable analysis techniques are needed as a first step in the development of synthesis techniques.

Below are presented some of the currant techniques employed in the analysis of nonlinear systems with stochastic inputs. Also mentioned are the limitations associated with each of these methods.

II. The Analysis of Nonlinear Control Systems with Random Inputs

According to R. C. Boston (1953), who first introduced the technique , the method of statistical linearization , is an inexact method of allowing systems with a certain class of nonlinearities, subject to random inputs, to be analyzed. The class of nonlinearities allowing the method of statistical linearlizations to be used are zero-memory nonlinearities in both open loop and feedback configurations.

Zero-memory nonlinearities are those nonlinearities with responses determined completely by the instantaneous amplitude of the input to the element. (The method does not apply to nonlinearities with memory, e.g., hysteresis).

The purpose of the analysis is to allow the computation of the probability denisty function (pdf.) of the output when the pdf of the input is known. The configuration and nonlinearities of the system must be known before the method of statistical linearization can be applied.

The method involves replacing the nonlinear element of the system by a linear element with a parameter. The parameter, the equivalent gain, Keg, is evaluated by statistical considerations. After the gain is evaluated the analysis of the whole system proceeds using the methods of linear systems.

The relationship between the output, y, and the input, x, for zero-memory nonlinearities can be expressed as $y = f(x)$. The method of' statistical linearization involves assuming the form of $y = K_{eg}\, x + x_H$ as the input-output relationship and neglecting x_H, the "distortion function". Keg is chosen by satisfying the criterion of minimizing the mean square error, $\overline{[y - K_{eg}\, x]^2}$.

The mean square error is determined by applying the fundamental theorem of expectation, as shown below:

$$M = \overline{[y - K_{eg} X]^2} = E\{[y - K_{eg} X]^2\} = E\{|f(x) - K_{eg} X|^2\} =$$

$$\int_{-\infty}^{\infty} [f(x) - K_{eg} X]^2 p_x(X)\, dx = \int_{-\infty}^{\infty} f^2(x)\, p_x(x)\, dx - 2K_{eg}\int_{-\infty}^{\infty} x f(x)\, p_x(x)\, dx$$

$$+ K_{eg}\int_{-\infty}^{\infty} x^2\, p_x(x)\, dx .$$

By differentiating M with respect to Keg and setting the result equal to zero an equation results which can be solved for the value of Keg that minimizes M.

$$\frac{\partial M}{\partial K_{eg}} = -2\int_{-\infty}^{\infty} x\, f(x)\, p_x(x)\, dx + 2\, K_{eg} \int_{-\infty}^{\infty} x^2 p_x(x)\, dx = 0$$

$$\therefore \quad K_{eg} = \frac{\int_{-\infty}^{\infty} x\, f(x)\, p_x(x)\, dx}{\int_{-\infty}^{\infty} x^2 p_x(x)\, dx}$$

The method of statistical linearization is especially well adopted to gaussian inputs since, after the conversion of the nonlinear system to an approximate linear system, the theory of linear systems assures that every variable in the system is gaussianly distributed and so, therefore, is the output.

Under the assumption of a gaussian input the expression for the computation of the Keg is greatly simplified. The pdf, of the input to the nonlinearity approximation is $p_x(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\frac{(x-\bar{x})^2}{\sigma_x^2}}$.

$$\int_{-\infty}^{\infty} x^2 p_x(x)\, dx = E(x^2) = \sigma_x^2 + (E[x])^2$$

$$\text{So} \qquad K_{eg} = \frac{\int_{-\infty}^{\infty} x\, f(x) \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\frac{(x-\bar{x})^2}{\sigma_x^2}}\, dx}{\sigma_x^2 + \bar{X}^2}$$

The nonlinearity is saw replaced by the linear systems with Keg, The resulting linear system is analyzed using the usual linear system analysis techniques.

According to Booton (1953), the above analysis is also applicable to servomechanisms and feedback control systems with unity feedback. The method of statistical linearization is not so straight forward for these configurations and leads to an analysis equivalent to the analysis of a linear system with a parameter, The result of the analysis indicates that multimodal behavior of the response is indicated.

This type of response is physically verified, but the method of statistical linearization does not predict when the response will be of a particular mode.

In a more recent work, Pervozvanskii (1965) has generalized the method of statistical linearization and extended it to nonlinear time-varying systems.

An example of statistical linearization is given in the appendix.

I. Approximate Analysis of Nonstationary Nonlinear Systems By Semiinvariants

The purpose of the analysis is to find the probability density function (pdf) of the output of a system which can be represented as a nonlinear, time-varying, differential equation of the form $\frac{dx(t)}{dt} = a(x(t),t) + b(t)\,w(t)$, where $w(t)$ is white gaussian noise.

As shown by M. L. Dashevskii (1966, 1967), the method involves a generalization of a technique that is very familiar from statistics. Recall the familiar technique of using the characteristic function, $M = E[e^{jzx}]$, to calculate the moments of the variables, $\overline{X^k} = (-j)^k \frac{\partial^k M}{\partial z^k}\Big|_{z=0}$. The function $\Psi = \ln M$, called the "second characteristic function", was used to compute the semi-invariants, $\lambda_k = (-j)^k \frac{\partial^k \Psi}{\partial z^k}\Big|_{z=0}$. (This method can be found in the exercises of Wozencraft and Jacobs' PRINCIPLES OF COMMUNICATIONS). The generalization involves extending the technique to the case when all the above quantities are functions of time, t. $\overline{X(t)^k} = (-j)^k \left.\frac{\partial^k M(t,z)}{\partial z^k}\right|_{z=0}$ and $\lambda_k(t) = (-j)^k \left.\frac{\partial^k \Psi(t,z)}{\partial z^k}\right|_{z=0}$ are the two generalizations upon which the method is based.

From the Fokker-Planck or Kolgmogorov's equation the partial differential equation for the pdf of the output $\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[p(x,t)a(x,t)] + \frac{1}{2}b \ldots \, \partial x^-$, is manipulated into an integro-partial differential equation in $M(z,t)$, $a(x,t)$, $p(x,t)$, and $b(t)$:

$$\frac{\partial M(z,t)}{\partial t} = jz \int_{-\infty}^{\infty} e^{jzx(t)} a(x(t),t) p(x(t),t)\, dx(t) - \frac{b^2(t)}{2} z^2 M(z,t).$$

From the equation $\Psi(z,t) = \ln M(z,t)$ the integro-partial differential equation involving $\Psi(z,t)$, $b(t)$, $a(x,t)$, and $p(x,t)$ are obtained: $\frac{\partial \Psi(z,t)}{\partial t} = e^{-\Psi(z,t)} jz \int_{-\infty}^{\infty} e^{jzx(t)} a(x(t),t) p(x(t),t)\, dx(t) - b^2(t) z^2/2$

Mow, differentiating the above with respect to z, multiplying by (-j), and setting $z = 0$.

Yields:
$$\left.\frac{\partial^2 \Psi(z,t)}{\partial z \partial t}\right|_{z=0} = j\, e^{-\Psi(z,t)} \int_{-\infty}^{\infty} [1 + z(jx - \Psi'(z,t))] \cdot a(x,t) e^{jzx} p(x,t)\, dx \Big|_{z=0} - b^2(t) z \Big|_{z=0}$$

Yielding:

$$\frac{d\lambda_1(t)}{dt} = (-j)\frac{\partial^2 \Psi(0,t)}{\partial z \partial t} = \int_{-\infty}^{\infty} a(x(t),t)\ p\ (x(t),t)\ dx(t).$$

By taking (-j) $\frac{\partial}{\partial z}$ to operate on the above yields:

$$\frac{d\lambda_2(t)}{dt} = \frac{\partial^3 \Psi(0,t)}{\partial z^2 \partial t} = \int_{-\infty}^{\infty} (x - \lambda_1)\ a\ (x(t),t) p(x(t),t) dx(t) + b^2(t).$$

Similarly the above operation is continued until $\frac{d\lambda_1(t)}{dt}$, $\frac{d\lambda_2(t)}{dt}$, $\frac{d\lambda_3(t)}{dt}$, $\frac{d\lambda_4(t)}{dt}$, and $\frac{d\lambda_5(t)}{dt}$ are obtained.

The $\lambda k(t)$ arc the semi-invariants and are of fundamental importance in the method. The still unknown p $(x(t),t)$ is expanded in an Edgeworth series which involves the unknown lower parameters $\lambda_1(t)$, $\lambda_2(t)$, $\lambda_3(t)$, $\lambda_4(t)$, and $\lambda_5(t)$. The above integro-differential equations are solved simultaneously for the $\lambda k(t)$, $k = 1,2,...,5$, and used in the Edgeworth series as a good representation of the pdf, $p(x(t),t)$, of the output.

Only a finite number of the $\frac{d\lambda_k(t)}{dt}$ are solved for simultaneously to keep the work load as low as possible. It is assumed that all $\lambda k(t) = 0$ for $k \geq 6$.

The power of this method lies in the fact that it easily handles the time-varying nonlinear differential equations thae have general nonlinearities involving time.

Owe of the main drawbacks of the method is that the number of integro-differential equations that must be solved simultaneously in order to obtain the required semi-invariants greatly. increases as the order of the differential equations describing the system increases.

IV. Volterra Functional Analysis of Nonlinear Tine-Varying Systems with Random Inputs

According to Y. H. Ku (1965, 1967), the recent Volterra functional method is extremely powerful in that it can be used to analyze systems that can be represented as differential equations that are nonlinear, time-varying, and that have dererministic or stochastic inputs.

Since the method of analysis for the case of stochastic inputs is very similar but Slightly wore complicated than the analysis for the case of deterministic inputs, the deterministic case will be discussed here first. The method is applicable when the systems can be represented by differential equations sf the following form:

i) $\mathcal{Z}(D)x(t) + F(x,\dot{x},\ldots,x^{(n-1)}) = r(t),\ D = \frac{d}{dt}$

ii) $\mathcal{Z}(D)x(t) + g(t)\ F(x,\dot{x},\ldots,x^{(n-1)}) = r(t)$

iii) $L(t;D)x(t) + F(x,\dot{x},\ldots,x^{(n-1)}) = r(t)$

iv) $L(t;D)x(t) + g(t)\ F(x,\dot{x},\ldots,x^{(n-1)}) = r(t)$

where $L(t;D)$ denotes a linear operator in both t and D; F is an analytic nonlinear function of the response, $x(t)$, and its time derivatives, $g(t)$ denotes a function of time, and $r(t)$ denotes the input, either deterministie or a sample function $\{r(t), -\infty < t < \infty\}$ from a strict sense stationary source with bounded moments of all orders.

For $\mathcal{Z}(D)\ x\ (t) + F(x,\dot{x},\ldots,x^{(n-1)}) = r(t)$ and a deterministic $r(t)$, the method yields a solution of the form $x(t) = \sum_{n=1}^{\infty} x_n(t)$,

$$x_n\ (t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_n\ (T_1,T_2,\ldots,T_n)r(t-T_1)\ldots r(t-T_n)dT_1\ldots dT_n$$

and where the kernel for the nth term is an n - dimensional kernel $h_n\ (t_1,\ t_2,\ldots,\ t_n)$. Thus, the method is a generalization of the convolution integral used in linear system analysis. Indeed; the first term of the series, $x_1(t)$, is simply a convolution sf the first input, $r(t)$, and the impulse $h_1(t)$ of the linear portion of the pverall system. Recurrence relations exist for computing the Volterra kernels in terns of previous kernels and previously computed terms of the series. These recurrence relations make the evaluation sf the several convolutions less tedious.

Only a finite number of terms of the series are required to closely approximate the nonlinear system, a situation analogous to the use of a finite number of terms sf a Fourier series to represent a function.

For $L(t;D)\, x(t) + F(x,\dot{x},\ldots,x^{(n-1)}) = r(t)$ and a deterministic $r(t)$, the method is very similar to the method for the case given above except that instead of using an impulse response $h_1(t)$ for calculating $x_1(t)$, there is a time-varying system function $k_1(t,T)$ such that $x_1(t) = \int_{-\infty}^{\infty} k_1(t,T)\, r(t-T)\, dT$. The $k_1(t,T)$ can be found by first finding $K_1(t,s)$ by Zadeh's method and inverse Laplace transforming.

The other two cases of a nonlinearity of the form of $g(t)$ $F(x,\dot{x},\ldots,x^{(n-1)})$ are treated in the same way as the above two cases.

The analysis of the four different dorms of differential equations for the case of a stochastic input deviates only slightly from the analysis for the case of a deterministic input. For the differential. equation of form i).

$$Z(D)\, x(t) + F(x,\dot{x},\ldots, x^{(n-1)}) = r(t) \text{ where } \{ r(t), -\infty < t < \infty \}$$

is a sample function from a strict sense stationary source with moments of all orders bounded, the linear systems is given by $Z(D)\, x_1(t) = r(t)$. The solution for the linear part is $x_1(t) = \int^{t} h_1(t-T)\, r(T)\, dT^{(1)}$, where $h_1(t)$ represents the impulse response of the linear portion system. Denotiug the ensemble average of $r(t)$ and $x(t)$ by $\langle r(t)\rangle_r$ and $\langle x(t)\rangle_x$ and applying these averages to both members of (1) gives $\langle x_1(t)\rangle_x = \int_0^t h_1(t-\tau)\langle r(\tau)\rangle_r d\tau$

$$= \langle r(t)\rangle_r \int_0^t h_1(t-\tau)\,u(\tau)\,d\tau = \langle r(t)\rangle_r \; x_{1u}(t)$$

where the interchange of expectation and integration has taken place and $x_{1u}(t)$ is the response of the system to a deterministic unit step function $U(t)$. For the higher order term $\langle x_i(t)\rangle_x =$ $\langle r^i(t)\rangle_r \; x_{iu}(t)$ where the $\langle r^i(t)\rangle_r$ is the ith moment of the input and x_{iu} is a result identical to the deterministic case with $r(t) = U(t)$.

The other forms ii, iii, and iv, of the nonlinear differential equations are treated in exactly the same way except tkat for $L(t;D)$ the system function is used. All follow the form $\langle x_i(t)\rangle_x =$ $\langle r^i(t)\rangle_r \, x_{iu}(t)$ with $\langle x(t)\rangle_x = \sum_{i=1}^{\infty} \langle x_i(t)\rangle_x$ being the final solution,

As Ku mentions, the method described above is less tedioas when digital computer programs are used in performing the convolutions and sumations.

An example using the Volterra functional analysis is given in the appendix.

V. Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems

Put forward by D. M. Detchmendy and R. Sridhar (1966) this method is one of the most promising in tkat it caw yield an estimate of X, $\hat{x}$, when $\dot{X} = g(x,t) + k(x,t)u$ and $y(t) = h(x,t) +$ (observation error) where U represents an unknown input.

This method uses a variational approach and then uses invariant imbedding equations to solve the resulting two point boundary value problem.

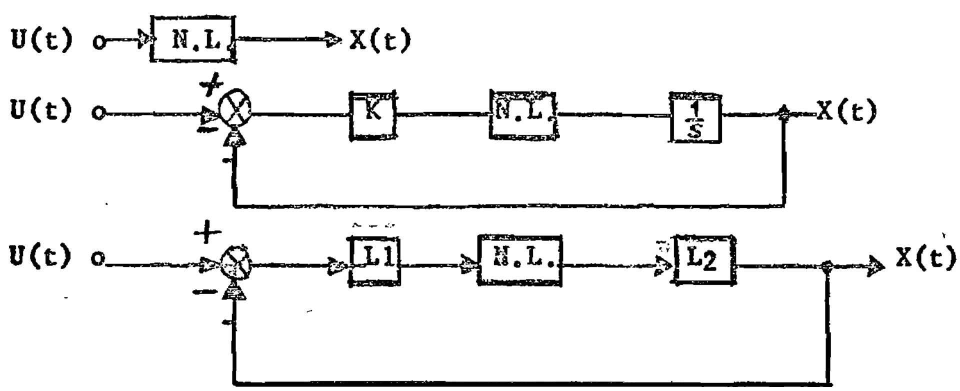
An important advantage is that the estimator obtained by this scheme can be implimented in real time.

There is also the possibility that this method can be used in conjunction with statistical linearization in helping to determine the gain for the feedback configuration, when, as mentioned above, the analysis proceeds as with a linear system with a parameter.

VI. Proposed Area of Research

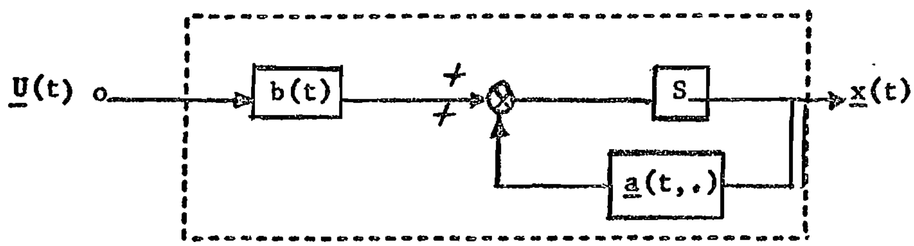
In summary, we have the following four methods or tools to apply to nonlinear systems with stochastic inputs:

(1.) The method of statistical linearization can be applied to control systems of the following configurations:



where U(t) is white gaussian noise and the nonlinearity, M.L. is of an acceptable type (zero-memory but time-varying or time-invariant). L_1 and L_2 in the above are linear systems.

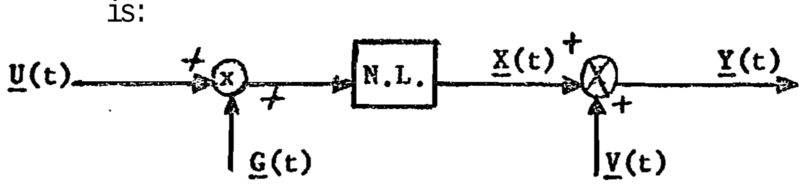
2.) The method of semi-invariant@ is applicable to the following configuration:



or any configuration that allows a nonlinear differential equation to be explicity written. Again, U(t) is gaussian white noise. The precision of the results using the method sf semi-invariants is much better than the method of statistical linearization and it can be used in the analysis of a larger class of nonlinear systems (i.e., time-varying nonlinearities).

(3.) The Volterra functional analysis method is applicable if the system can be formulated as a differential equation with a linear part and a nonlinear part of the form $g(t)\ F(x,\dot{x},\ldots,x^{(n-1)})$. The only approximation involved in this method is that of using only a finite number of the terms in the series. The random inputs must be strict-sense stationary and have all movements bounded.

(4.) The sequential estimation method is good in that it can be used when the input is a composite of control function and noise. The allowed system configuration is:



where U(t) and V(t) are unknown noises and G(t) is a deterministic control force.

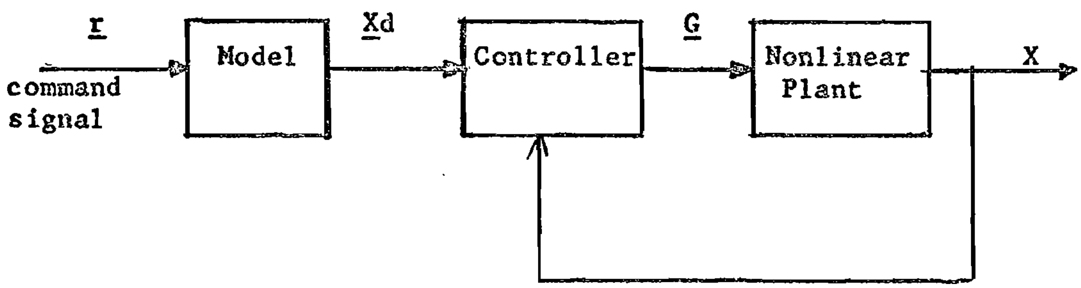
The response of a linear system subjected to stochastic inputs and a deterministic control force can be obtained by adding the response of the system due to the stochastic inputs alone to the response of the system due to the deterministic control force alone. That the sum of the responses to the individual inputs is the responses of the system to the combined inputs follows from the validity of the superposition principle for linear systems.

A common characteristic of three of the four approaches to the analysis of nonlinear systems subjected to random inputs given above was the absence of a deterministic control force. No such convient superposition principle exists for nonlinear systems so that, in general, the response of a nonlinear system to a stochnstfc input and a deterministic input is not the sum of the response due to the stochastic part alone and the response due to the deterministic part alone. Because superposition is not valid for nonlinear systems, the response to stochastic inputs alone is of no value in trying to determine the response to stochastic inputs and a deterministic control.

In the proposed research, it is my desire to attempt to solve the problem sf analysis of nonlinear systems subjected to stochastic inputs and deterministic control. I would also like to consider the problem of filtering under the above conditions when the observation is contaminated by measurement noise. My approach will be to learn how to derive the Stratonovich-Kushner-

Bucy fittering equation for this general nonlinear system having stochastic and deterministic imputs and to learn the Ito and Stratonavich's approach to manipulating this equation. I would then try to generalize the four approaches mentioned above to this problem.

Another approach to the problem that I plan to take in the proposed research will be in following the lead of a recent paper by H. F. VanLandingham and W. A. Blàckwell (1967) on the design of a technique that generates a control signal which forces tho state of a nonlinear plant to be close to the state of a reference model by ingeniously applying Liapunov's second method. The system configuration is shown below.



The model used is a linear antonomous system and represents "ideal" system behavior, The state variable representation of this model is $\dot{\underline{Xd}} = Ao\, \underline{Xd} + Bo\, \underline{r}$, where Xd is the state of the linear constant system model, $\underline{r}$ is the input vector, and A6 and Bo are constant matrices.

The actural nonlinear system is characterized by the nonlinear differential equation $\dot{\underline{X}} = f(\underline{x},\underline{G},t)$, where $\underline{X}$ is the actual state, $\underline{G}$ is the deterministic control vector, and t is time, The error, $\underline{e}$, is defined as $\underline{X}d - \underline{X}$ and can be conviently manipulated into the differential equation $\dot{\underline{e}} = Ao\ \underline{e} + Ao\underline{X} - f(\underline{x}, \underline{G}, t) + Bo\ \underline{r}$.

Liapunov's second method is applied to this differential equation by assuming a form for the Liapunov function of $V = \underline{e}^T P \underline{e}$, where P is a symmetric positive definite matrix. The derivative of V with respect to time is

$$\dot{V} = \underline{e}^T \left[Ao^T P + PAo \right] \underline{e} + \left\{ 2\underline{e}^T \left[Ao\underline{X} - f(\underline{x}, \underline{G}, t) + Bo\ \underline{r} \right] \right\}$$

h deterministic control $\underline{G}$ is synthesized that makes V negative definite. This condition ensures that the error, $\underline{e}$, is asymptotically stable and, hence, goes to zero.

My proposed research on this aspect of the problem will be to investigate the possibility sf replacing the Liapunov funtion of the deterministic case with a "stochastic Liapunov function" in synthesizing a $\underline{G}$ to ensure $\underline{e}$ to be asymptotically stable 'and thus generalize the above method to the control of nonlinear systems with random inputs.

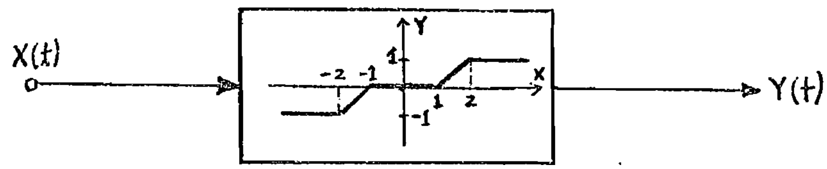
The use of stochastic Liapunov functions has received a good deal of attention in the last three years. Kushner (1967) has done quite a bit in the area of stochastic Liapunov functions.

In working with stochastic Liapunov functions, I realize that this part of the research will be less practical-oriented since, as Kushner (1965) mentioned, a shortcoming common to both deterministic and stochastic Liapunov functions is the difficulty of finding them.

VII. Appendix

Statistical Linerazation Example

The figure below shows a nonlinear element representing saturation with a dead zone.



In equation form, $Y(t) = f\left[X(t)\right]$, where X(t) is a zero mean gaussian random process with unit variance and having a pdf of $P_x(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} X^2(t)\right)$.

The assumption made is that $X_H(t)$ is neglizible in the representation $Y(t) = Keg\, X(t) + X_H(t)$. Therefore $X_H(t)$ is dropped, yielding $\hat{Y}(t) = Keg\, X(t)$.

The mean squared error which is to be minimized as the criterion satisfied by this technique is $\left[Y(t) - \hat{Y}(t)\right]^2 = \left[Y(t) - Keg\, X(t)\right]^2$

$$= \left[f(X(t)) - Keg\, X(t)\right]^2.$$

So Keg is to be chosen to minimize

$$M = \int_{-\infty}^{\infty} \left[Y(t) - \hat{Y}(t)\right]^2 p_x(x)\,dx$$

The general result for Keg, shorn earlier, is

$$Keg = \frac{\int_{-\infty}^{\infty} x\, f(x)\, p_x(x)\,dx}{\int_{-\infty}^{\infty} x^2\, p_x(x)\,dx}.$$

Since $p_x(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2(+1)\right)$,

$$\int_{-\infty}^{\infty} x^2 p_x(x)\,dx = \int_{-\infty}^{\infty} x^2 \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = \sigma^2 - (\mu_x)^2 = 1$$

In the above $\mu_x = 0$ since $\mu_x = \int_{-\infty}^{\infty} x\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = -\int_{-\infty}^{\infty} x'\frac{e^{-\frac{(x')^2}{2}}}{\sqrt{2\pi}}\,dx' = -\mu_x$

under $x' = -x$

and the only quantity equal to its negative is zero.

Evaluating the expression for Keg in terms of the appropriate value of f(x) yields:

$$\text{Keg} = \Big[\int_{-\infty}^{-2} x(-1)\,p_x(x)\,dx + \int_{-2}^{-1} x(1+x)\,p_x(x)\,dx + \int_{-1}^{+1} x\cdot 0\cdot p_x(x)\,dx$$
$$+ \int_{1}^{2} x(x-1)\,p_x(x)\,dx + \int_{2}^{\infty} x(1)\,p_x(x)\,dx\Big]/1.$$

Substituting in the pdf yields:

$$\text{Keg} = \Big[\int_{-\infty}^{-2} x(-1)\,e^{-\frac{x^2}{2}}\,dx + \int_{-2}^{-1} (x+x^2)\,e^{-\frac{x^2}{2}}\,dx + \int_{-1}^{+1} 0\,dx$$
$$\int_{+1}^{2} (x^2-x)\,e^{-\frac{x^2}{2}}\,dx + \int_{2}^{\infty} x\,e^{-\frac{x^2}{2}}\,dx\Big]/\sqrt{2\pi}$$

Using symmetry properties of even functions and shuffling the integrals around yields:

$$\text{Keg} = 2\Big[\int_{2}^{\infty} (x^2-x)\,e^{-\frac{x^2}{2}}\,dx + \int_{1}^{2} x\,e^{-\frac{x^2}{2}}\,dx\Big]/\sqrt{2\pi}.$$

To further simplify the above expression for Reg three integrations by parts are carried out below. Use is made of the error function.

$$i.)\ \int_{2}^{\infty} x\,e^{-\frac{x^2}{2}}\,dx = \frac{1}{2}\int_{4}^{\infty} e^{-\frac{y}{2}}\,dy = -e^{-\frac{y}{2}}\Big|_{4}^{\infty} = -e^{-\infty} + e^{-2} = e^{-2}.$$

$$ii.)\ \int_{1}^{2} x^2\,e^{-\frac{x^2}{2}}\,dx = \int_{1}^{2} -x\left(-e^{-\frac{x^2}{2}}x\,dx\right) = -x\,e^{-\frac{x^2}{2}}\Big|_{1}^{2} + \int_{1}^{2} e^{-\frac{x^2}{2}}\,dx$$
$$= -2e^{-2} + e^{-\frac{1}{2}} + \sqrt{\tfrac{\pi}{2}}\cdot\frac{2}{\sqrt{\pi}}\int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}} e^{-u^2}\,du = -2e^{-2} + e^{-\frac{1}{2}} + \sqrt{\tfrac{\pi}{2}}\left[\operatorname{erf}(\sqrt{2}) - \operatorname{erf}\left(\tfrac{1}{\sqrt{2}}\right)\right]$$

$$iii.)\ \int_{1}^{2} -x\,e^{-\frac{x^2}{2}}\,dx = e^{-\frac{x^2}{2}}\Big|_{1}^{2} = e^{-2} - e^{-\frac{1}{2}}.$$

Resubstituting into the expression for Keg gives

$$\text{Keg} = 2\left[e^{-2}-2e^{-2}+e^{-\frac{1}{2}}+\sqrt{\tfrac{\pi}{2}}\left\{erf(\sqrt{2})-erf(\tfrac{1}{\sqrt{2}})\right\}+e^{-2}-e^{-\frac{1}{2}}\right]/\sqrt{2\pi}$$

$$= \left[erf(\sqrt{2})-erf(\tfrac{1}{\sqrt{2}})\right]$$

The error function erf(·) is well tabulated.

From the expression for Keg, the expression for the estimate of the output,

$$\hat{Y}(t) = \text{Keg}\, X(t) = \left[erf(\sqrt{2})-erf(\tfrac{1}{\sqrt{2}})\right] X(t),$$

is obtained.

Volterra Functional Analysis Example

$$t^2\frac{d^2y}{dt^2}+4t\frac{dy}{dt}+2y+\mu y^2 = r(t)$$

r(t) is a stochastic imput having all moments $\langle r^i(t)\rangle_r, i = 1,2,3,\ldots,$ known and bounded.

First, consider the method of Volterra functional analysis applied to the deterministic case of r(t) = U(t), a unit step, since the results of this analysis are required in evaluating the case of a stochastic r(t).

The above ordinary nonlinear differential equation is of the form $L(t,D)Y(t) + F(y,\dot{y},\ldots,Y^{(n-1)}) = r(t)$, so the method of Volterra functional analysis is applicable.

In order to start the Volterra functional analysis procedure it is necessary to obtain the impulse response from the linear portion of the nonlinear differential equation.

Finding the impulse response involves applying the variation of parameters method to the linear portion of the example. The differential equation now being considered is $t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2Y = F(t)$. To apply the method of variation of parameters it is first necessary to have the complementary or transient solution which is the solution of the associated homogeneous differential equation, $t^2 \frac{d^2y}{dt^2} + 4t \frac{dy}{dt} + 2Y = 0$. The associated differential equation in this example is seen to be of the Canchy type, ameanable to solution by the standard trick of a change of independent variable using the substitution $t = e^z$. (Another method of obtaining the complementary solution would be to use Mellin transforms.)

The substitution requires evaluation of:

$$t = e^z \implies z = \ln t$$

$$\frac{dt}{dz} = e^z = t \implies \frac{dz}{dt} = e^{-z} = \frac{1}{t}$$

$$\frac{dy}{dt} = \frac{dy}{dz}\cdot\frac{dz}{dt} = \frac{dy}{dz}\cdot\frac{1}{t} = \frac{1}{t}\frac{dy}{dz}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dz}\left[\frac{dy}{dt}\right]\cdot\frac{dz}{dt} = \frac{d}{dz}\left[\frac{1}{t}\frac{dy}{dz}\right]\cdot\frac{1}{t}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dz}\left[\frac{1}{t}\frac{dy}{dz}\right]\frac{1}{t} = \left[\frac{1}{t}\frac{d^2y}{dz^2} - \frac{\frac{dt}{dz}}{t^2}\frac{dy}{dz}\right]\cdot\frac{1}{t} = \frac{1}{t}\left[\frac{1}{t}\frac{d^2y}{dz^2} - \frac{1}{t}\frac{dy}{dz}\right]$$

$$= \frac{1}{t^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right].$$

Then applying the substitution to the associated homogeneous differential equation transforms the Canchy type differential equation into a linear, constant coefficient, ordinary differential equation as shown below:

$$t^2 \left(\frac{1}{t^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz} \right] \right) + 4t \left(\frac{1}{t}\frac{dy}{dz} \right) + 2Y = 0$$

or $D^2Y + 3DY + 2Y = 0$

or $(D + 2)\cdot(D + 1)Y = 0$

From the theory of linear constant coefficient differential equuations the complementary solution is $Y(z) = C_1 e^{-2z} + C_2 e^{-z}$, where C_1 and C_2 are arbitary constants. Reversing the substitution used above, replacing by LNt yields $Y(t) = C_1 e^{-2\ln t} + C_2 e^{-\ln t} = \frac{C_1}{t^2} + \frac{C_2}{t}$. That the complementary solution, $\frac{1}{t^2}$ and $\frac{1}{t}$, are independent can be checked by observing that the Wronskian is nonzero:

$$W(z) = \begin{vmatrix} \frac{1}{z^2} & \frac{1}{z} \\ \frac{-2}{z^3} & \frac{-1}{z^2} \end{vmatrix} = -\frac{1}{z^4} + \frac{2}{z^4} = \frac{1}{z^4} \neq 0$$

The linear differential equation $t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2Y = F(t)$ is of the form $a_n(t)\, D^nY + a_{n-1}(t)\, D^{n-1}Y + \ldots + a_0\, Y(t) = A(D,t)Y = F(t)$ and by being of this form has a particular solution, obtained by variation of parameters, of the form

$$Y(t) = \int_0^t F(z) \left[\frac{1}{a_n(z)W(z)} \sum_{i=1}^{n} y_i(t) W_{ni}(z) \right] dz,$$

where the Yi's are the n independent solutions of the associated homogeneous differential equation, $W(z)$ is the Wronskian, and the

$W_{ni}(\xi)$ are the ni-th cofactor of the Wronskian. The Wronskian is given by

$$= \begin{vmatrix} Y_1(\xi) & Y_2(\xi) & \cdots & Y_n(\xi) \\ \dot{Y}_1(\xi) & \dot{Y}_2(\xi) & \cdots & \dot{Y}_n(\xi) \\ \cdots & \cdots & \cdots & \cdots \\ Y_1^{(n-1)}(\xi) & Y_2^{(n-1)}(\xi) & \cdots & Y_n^{(n-1)}(\xi) \end{vmatrix}$$

The quantity in brackets above is known as the one-sided Green's function and is denoted by $g(t,\xi) = \dfrac{1}{a_n(\xi)\, W(\xi)} \sum_{i=1}^{n} Y_i(t)\, W_{ni}(\xi)$. The Green's function satisfies the useful property that the particular solution is given by $Y(t) = \int_0^t F(\xi)\, g(t,\xi)\, d\xi$.

For a differential equation of the type considered in this example the impulse response and the one-sided Green's function are identical. Therefore $h(t,\xi) = \begin{cases} g(t,\xi) & \text{for } 0 < \xi < t \\ 0 & \text{for } t < \xi \end{cases}$

For the present example, $W(\xi) = \begin{vmatrix} \frac{1}{\xi^2} & \frac{1}{\xi} \\ -\frac{2}{\xi^3} & -\frac{1}{\xi^2} \end{vmatrix} = \frac{1}{\xi^4}$

and $\sum_{i=1}^{2} Y_i(t)\, W_{ni}(\xi) = \begin{vmatrix} \frac{1}{\xi^2} & \frac{1}{\xi} \\ \frac{1}{t^2} & \frac{1}{t} \end{vmatrix} = \frac{1}{\xi^2 t} - \frac{1}{t^2 \xi} = \frac{(t-\xi)}{\xi^2 t^2}$

So $h(t,\xi) = g(t,\xi) = \dfrac{1}{a_n(\xi)\, W(\xi)} \sum_{i=1}^{2} Y_i(t)\, W_{ni}(\xi) = \dfrac{1}{\xi^2\left(\frac{1}{\xi^4}\right)} \left[\dfrac{t-\xi}{\xi^2 t^2}\right] = \begin{cases} \frac{(t-\xi)}{t^2}, & t \geq \xi \\ 0, & t < \xi \end{cases}$

The impulse response or Zadeh's system function could have been obtained by Zadeh's method. The above approach was taken here because of the general familiarity of the several techniques used.

The techniques used in this example can be found in DeRusso or any good differential equations book or book on the techniques of mathematical physics.

Returning now to the application of the Volterra functional analysis technique to the solution of the nonlinear differential equation $t^2 \frac{d^2y}{dt^2} + 4t\frac{dy}{dt} + 2y + \mu y^2 = r(t)$ with a linear portion impulse response represented using a notation in keeping with that suggested in DeRusso of $h*(t,t-z) = h(t,z) = \begin{cases} \frac{(t-z)}{t^2}, & t \geq z \\ 0, & t < z. \end{cases}$

The linear system is given by the convolution integral

$$Y_1(t) = \int_0^t h_*(t,t-z)\, r(z)\, dz.$$

The quadratic and cubic systems are

$$Y_2(t) = -\mu \int_0^t h_*(t,t-z)\, Y_1^2(z)\, dz.$$

$$Y_3(t) = -2\mu \int_0^t h_*(t,t-z)\, Y_1(z)\, Y_2(z)\, dz.$$

In general, the higher order system is given by

$$Y_i(t) = -\mu \sum_{j=1}^{i-1} \int_0^t h_*(t,t-z)\, Y_{i-j}(z)\, Y_j(z)\, dz, \quad i = 4,5,6,\ldots$$

For this example, $r(t) = U(t)$, a unit step function, so the systems reduce to:

$$Y_1(t) = \int_0^t \frac{(t-z)}{t^2} U(z)\, dz = \int_0^t \frac{(t-z)}{t^2} dz = \frac{tz - \frac{z^2}{2}}{t^2}\Big|_0^t = \frac{t^2 - \frac{t^2}{2}}{t^2} = \frac{1}{2}$$

$$Y_2(t) = \int_0^t (-\mu) \frac{(t-z)}{t^2} \left(\tfrac{1}{2}\right)^2 dz = -\frac{\mu}{4}\left(\tfrac{1}{2}\right) = -\frac{\mu}{8}$$

$$Y_3(t) = -2\mu \int_0^t \frac{(t-z)}{t^2} \left(\tfrac{1}{2}\right)\left(-\tfrac{\mu}{8}\right) dz = \frac{\mu^2}{8} \cdot \frac{1}{2} = \frac{\mu^2}{16}$$

$$\cdots \qquad \cdots$$

$$Y_i(t) = -\mu \sum_{j=1}^{i-1} \int_0^t \frac{(t-z)}{t^2}\, Y_{i-j}(z)\, Y_j(z)\, dz, \quad i = 4,5,6,\ldots$$

Now that the analysis of the nonlinear system for the case of a deterministic unit step input has been conpleted above, it is permissible to proceed with the analysis of the case of a stochastic input since all the information accrued in the analysis for the unit step will be needed. Applying ensemble averages to the linear portion convolution integral yields

$$\langle y_1(t)\rangle_y = \langle \int_0^t h_*(t, t-z)\, r(z)\, dz \rangle_r = \int_0^t h_*(t,t-z) \langle r(z)\rangle_r\, dz = \langle r(t)\rangle_r \int_0^t h_*(t,t-z)\, U(z)\, dz = \langle r(t)\rangle_r\, y_{1u}(t),$$

where $Y_{1u}(t)$ is notation indicating that it is the deterministic response when the input is a unit step. In general, $\langle Y_i(t)\rangle_y = \langle r^i(t)\rangle_r \; X_{iu}(t)$ for $i = 1,2,3,\ldots$

For the above example:

$$\langle Y_1(t)\rangle_y = \tfrac{1}{2}\langle r(t)\rangle_r$$

$$\langle Y_2(t)\rangle_y = -\tfrac{\mu}{8}\langle r^2(t)\rangle_r$$

$$\langle Y_3(t)\rangle_y = \tfrac{\mu^2}{16}\langle r^3(t)\rangle_r$$

$$\cdots \qquad \cdots$$

$$\langle Y_i(t)\rangle_y = -\mu \sum_{j=1}^{i-1} \int_0^t \frac{(t-z)}{t^2}\, y_{i-j}(z)\, y_j(z)\, dz \,\langle r^i(t)\rangle_r$$

And the final solution is

$$\langle Y(t)\rangle_y = \sum_{i=1}^{\infty} \langle r^i(t)\rangle_r \; Y_{iu}(t) = \tfrac{1}{2}\langle r(t)\rangle_r - \mu/8\,\langle r^2(t)\rangle_r + \mu^2/16\,\langle r^3(t)\rangle_r - \ldots$$

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