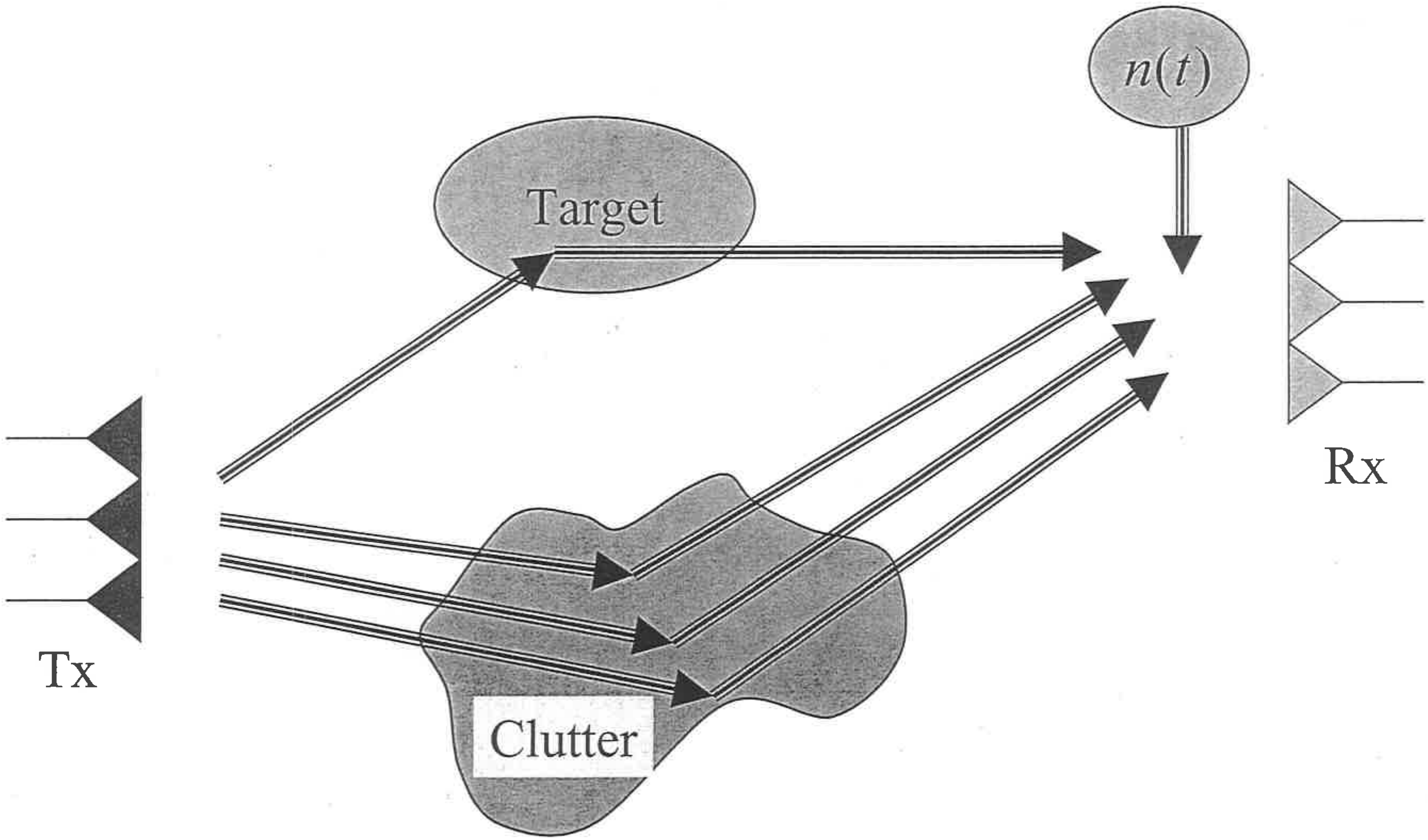
Optimized Multichannel Waveforms with Application to Polarimetrics

S. U. Pillai
Electrical Engineering Dept
Polytechnic University
Brooklyn, New York 11201

J. R. Guerci
DARPA/SPO,
3701 North Fairfax Dr.
Arlington, VA 22203

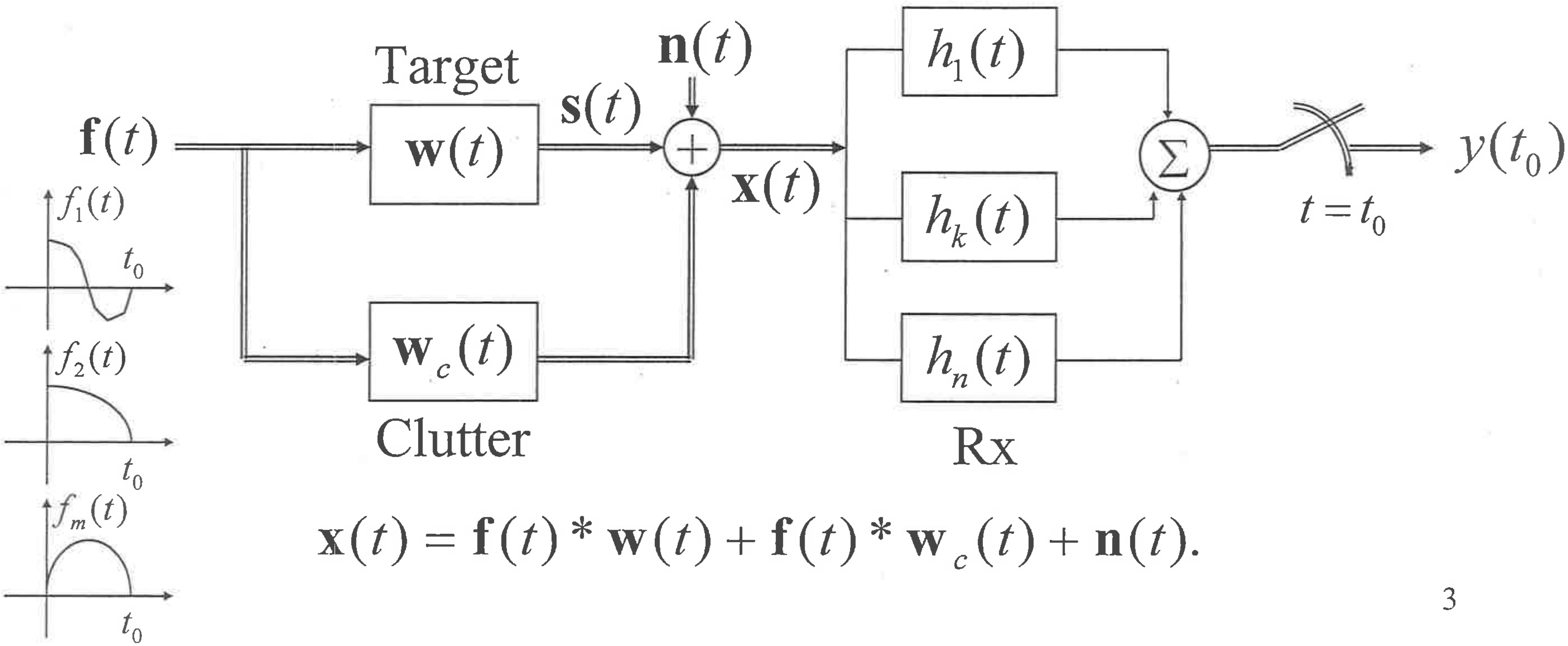
Multichannel Matched Transmit-Receiver Design in Presence of Signal Dependent Interference and Noise



- Multichannel matched illumination problem in presence of clutter.
- Transmitted signal vector excites both the desired multicahnnel target and clutter (undesired target).

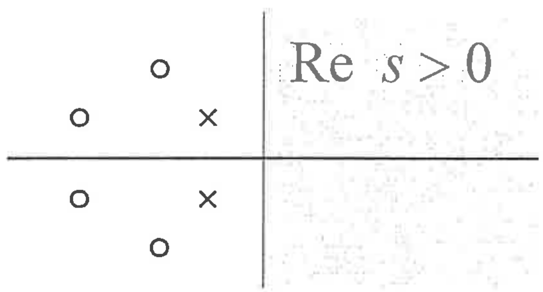
Problem:

- Optimize the transmit waveform vector and design a receiver structure to maximize the multichannel target return and minimize the clutter return in presence of noise.



$$\mathbf{x}(t) = \mathbf{f}(t) * \mathbf{w}(t) + \mathbf{f}(t) * \mathbf{w}_c(t) + \mathbf{n}(t).$$

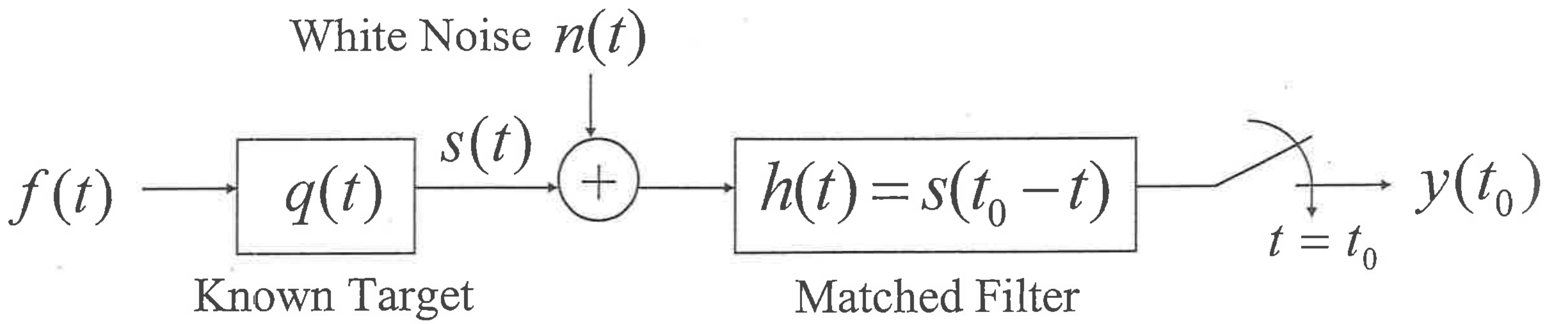
- Communication Scene: Maximize the direct path return and minimize multipath return reflections by optimizing the transmit signal and matched receiver. Also useful in underwater scenario.
- A Key Result (Scalar case): All optimum transmit solutions with or without clutter must be minimum phase waveforms (i.e., $f(t) \leftrightarrow F(s)$, then $F(s)$ has no poles and zeros in the right half plane.)



Poles and Zeros of a Minimum Phase Function

Remark: $f(t)$ = conventional chirp pulse is not minimum phase;

- Scalar case: Solution in the absence of clutter



$$s(t) = f(t) * q(t) = \int_0^{t_0} f(\tau) q(t-\tau) d\tau \, .$$

Output SNR at t_0 :

$$\rho(t_0) = \frac{1}{N_0} \int_0^{t_0} | s(t) |^2 \, dt.$$

Maximize $\rho\,(t_0)$ subject to

$$\int_0^{t_0} | f(t) |^2 \, dt = 1.$$

$$\rho(t_0) = \frac{1}{N_0}\int_0^{t_0}|s(t)|^2\,dt = \frac{1}{N_0}\int_0^{t_0} f(\tau_1)\psi(\tau_1)\,d\tau_1$$
$$\leq \frac{\lambda}{N_0}\int_0^{t_0}|f(\tau_1)|^2\,d\tau_1 = \frac{\lambda}{N_0}. \qquad (1)$$

Target Kernel: $K(\tau_1,\tau_2) = \int_0^{t_0} q(t-\tau_1)q(t-\tau_2)dt \geq 0$,
with equality in (1) iff

$$\psi(\tau_1) = \int_0^{t_0} K(\tau_1,\tau_2)f(\tau_2)d\tau_2 = \lambda\, f(\tau_1), \quad 0 \leq \tau_1 \leq t_0$$

$$\rho_{\max}(t_0) = \lambda_{\max}(t_0)/N_0.$$

Desired $f(t)$ = Eigenfunction associated with $\lambda_{\max}(t_0)$.
(Necessarily minimum phase!)

- Applications:
  - Joint Time-Bandwidth Optimization.
    (Finite pulse that maximizes energy in a desired band)
  - Multiple BW constraints is analogous to the presence of clutter.

General Case (Scalar):

Output SNR:

$$\rho_f(t_0) = \frac{\left(\int_0^{\infty} h(\tau)s(t_0-\tau)d\tau\right)^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty}|H(\omega)|^2\left(G_n(\omega)+G_c(\omega)|F(\omega)|^2\right)d\omega}.$$

Define

$$|L_f(j\omega)|^2 = G_n(\omega)+G_c(\omega)|F(\omega)|^2.$$

$L_f(s)$: Wiener-Hopf Factor; analytic together with its inverse in $Re\ s > 0$. $L_f(s)$ depends on unknown $f(t)$.

$$\rho_f(t_0) = \frac{\left|\frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)S(\omega)e^{j\omega t_0}d\omega\right|^2}{\frac{1}{2\pi}\int_{-\infty}^{+\infty}|H(\omega)L_f(j\omega)|^2 d\omega} \le \frac{1}{2\pi}\int_{-\infty}^{+\infty}|K(\omega)|^2 d\omega$$

$$g(t) \leftrightarrow L_f^{-1}(j\omega)Q(\omega)F(\omega); \qquad g(t_0-t)u(t) \leftrightarrow K(\omega)$$

Hence

$$\max \rho_f(t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |K(\omega)|^2 \, d\omega = \int_{-\infty}^{+\infty} g^2(t_0 - t)u(t)dt$$

$$= \int_0^{t_0} g^2(t)dt \qquad (2)$$

with equality iff

$$H(\omega) = \mu\, L_f^{-1}(j\omega)K(\omega)$$

$$= \mu\, L_f^{-1}(j\omega)\left\{ L_f^{*\,-1}(j\omega)S^*(\omega)e^{-j\omega t_0} \right\}_+ . \qquad (3)$$

$h(t) \leftrightarrow H(\omega)$ is causal (generalized M.F.)

- For a given $f(t)$, the optimum causal M.F. in presence of clutter is given by (3). Corresponding realizable SINR in that case is given by the right side of (2).

$$\rho_{\max}(t_0) \triangleq \max \rho_f(t_0) = \int_0^{t_0} g^2(t)\,dt$$

where

$$g(t) \leftrightarrow L_f^{-1}(j\omega)S(\omega) = L_f^{-1}(j\omega)W(\omega)F(\omega),$$

and

$$|L_f(j\omega)|^2 = G_n(\omega) + G_c(\omega)\,|F(\omega)|^2.$$

- If $f(t)$ is causal, $g(t)$ is also causal, and $\rho_{\max}(t_0)$ is a function of $f(t)$. Problem here is to maximize $\rho_f(t_0)$, subject to

$$\int_0^{t_0} f^2(t)\,dt = E.$$

- Unlike the white noise case and in the absence of clutter, the solution for $f(t)$ here is not scalable.
- Nonlinear optimization problem, since $L_f(j\omega)$ depends on the desired unknown pulse $f(t)$. Iterative converging solution.

Multichannel Formulation

$$\mathbf{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_m(t) \end{pmatrix} \Leftrightarrow \mathbf{F}(\omega) = \begin{pmatrix} F_1(\omega) \\ F_2(\omega) \\ \vdots \\ F_m(\omega) \end{pmatrix}.$$

Target Transfer Matrix ($m \times m$):

$$\mathbf{w}(t) \leftrightarrow \mathbf{W}(\omega) = \left[w_{ij}(\omega) \right]_{ij}^{m}$$

Clutter: Stochastic frequency response ($m \times m$)

$$\mathbf{w}_c(t) \leftrightarrow \left[\mathbf{W}_{c,1}(\omega)\ \mathbf{W}_{c,2}(\omega) \cdots \mathbf{W}_{c,m}(\omega) \right]$$

$W_{c,k}(\omega)$ is the clutter response induced by $F_k(\omega)$.

Different clutter responses can be correlated.

Clutter Covariance Spectrum

$$\mathbf{G}(\omega)=\begin{pmatrix} G_{11}(\omega) & G_{12}(\omega) & \cdots & G_{1m}(\omega) \\ G_{21}(\omega) & G_{22}(\omega) & \cdots & G_{2m}(\omega) \\ \vdots & \vdots & \ddots & \vdots \\ G_{m1}(\omega) & G_{m2}(\omega) & \cdots & G_{mm}(\omega) \end{pmatrix} > 0, \quad (m^2 \times m^2)$$

where $G_{ij}(\omega) = E\left[\mathbf{w}_{c,i}(\omega)\mathbf{w}^*_{c,j}(\omega)\right], \ (m \times m).$

Received signal

$$\mathbf{x}(\omega) = \mathbf{w}(\omega)\mathbf{F}(\omega) + \sum_{i=1}^{m} \mathbf{w}_{c,i}(\omega) F_i(\omega) + \mathbf{n}(\omega).$$

Total Clutter Spectrum

$$\mathbf{G}_F(\omega) = \sum_{i=1}^{m}\sum_{j=1}^{m} F_i(\omega) F_j^*(\omega) G_{ij}(\omega).$$

Output SNR

$$\rho_o(t_0) = \frac{\left| \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{H}'(\omega)\mathbf{W}(\omega)\mathbf{F}(\omega) e^{j\omega t_0} d\omega \right|^2}{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{H}'(\omega)\{\mathbf{G}_F(\omega) + \mathbf{G}_n(\omega)\}\overline{\mathbf{H}}(\omega) d\omega}$$

$$\mathbf{G}_0(\omega) = \mathbf{G}_F(\omega) + \mathbf{G}_n(\omega) = \mathbf{L}(j\omega)\mathbf{L}^*(j\omega).$$

L(s): Left-Wiener factor of the 'total' spectrum.

$$\mathbf{g}(t) \leftrightarrow \mathbf{L}^{-1}(j\omega)\mathbf{W}(\omega)\mathbf{F}(\omega)$$

$$\phi(t) \leftrightarrow \mathbf{L}^{-1}(j\omega)\mathbf{W}(\omega)$$

$$\mathbf{g}(t_0 - t)u(t) \leftrightarrow \mathbf{K}(\omega)$$

Then

$$\rho_o(t_0) = \frac{1}{2\pi} \frac{\left| \int_{-\infty}^{+\infty} \mathbf{H}'(\omega)\mathbf{L}(j\omega)\mathbf{K}^*(j\omega)\,d\omega \right|^2}{\int_{-\infty}^{+\infty} \mathbf{H}'(\omega)\mathbf{L}(j\omega)\mathbf{L}^*(j\omega)\overline{\mathbf{H}}(\omega)\,d\omega}$$

$$\le \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathbf{K}^*(j\omega)\mathbf{K}(j\omega)\,d\omega = \int_0^{t_0} \mathbf{g}^*(t)\mathbf{g}(t)\,dt$$

with equality iff

$$\mathbf{H}_{opt}(\omega) = \{\mathbf{L}'(j\omega)\}^{-1}\mathbf{K}(j\omega).$$

With

$$\mathbf{g}(t) = \int_0^{t_0} \phi(t-\tau)\mathbf{f}(\tau)d\tau$$

and

$$\Omega(\tau_1, \tau_2) = \int_0^{t_0} \phi^*(t-\tau_1)\phi^*(t-\tau_2)dt$$

$$\rho_o(t_0) = \int_0^{t_0} \mathbf{g}'(t)\mathbf{g}(t)dt = \int_0^{t_0}\int_0^{t_0} \mathbf{f}'(\tau_1)\Omega(\tau_1,\tau_2)\mathbf{f}(\tau_2)d\tau_1 d\tau_2$$
$$\leq \lambda_{1,\max}$$

where $\lambda_{1,\max}$ corresponds to the largest eigenvalue of the equation

$$\int_0^{t_0} \Omega(\tau_1,\tau_2)\mathbf{f}(\tau_2)d\tau_2 = \lambda\mathbf{f}(\tau_1), \qquad (4)$$

and $\phi_1(t) = \mathbf{f}(t)$ corresponding to $\lambda_{1,\max}$ in (4) represents the optimum transmit vector.

- Highly nonlinear problem to maximize $\rho(t_0)$ over $\mathbf{f}(t)$.
- Iterative solution for $\mathbf{f}(t)$:

At stage k, with $\mathbf{f}_k(t) \leftrightarrow F_k(\omega)$ satisfying the constant energy condition

$$\int_0^{t_0} \mathbf{f}_k'(t)\mathbf{f}_k(t)dt = E, \quad \text{all} \quad k, \tag{5}$$

find the minimum phase solution of

$$\mathbf{L}_k(j\omega)\mathbf{L}_k'(j\omega) = \mathbf{G}_n(\omega) + \sum_{i=1}^{m}\sum_{j=1}^{m} \mathbf{G}_{ij}(\omega)F_{k,i}(\omega)F_{k,j}^*(\omega). \tag{6}$$

Let

$$\phi_k(t) \leftrightarrow \{\mathbf{L}_k'(j\omega)\}^{-1}\mathbf{Q}(\omega) \tag{7}$$

and compute

$$\Omega_k(\tau_1, \tau_2) = \int_0^{t_0} \phi_k(t-\tau_1)\,\phi_k(t-\tau_2)dt. \tag{8}$$

Find the largest eigenvalue $\lambda_1^{(k)}$ and the corresponding eigenfunction $\psi_1^{(k)}(t)$ of

$$\int_0^{t_0} \Omega_k(\tau_1, \tau_2)\psi_1^{(k)}(\tau_2)d\tau_2 = \lambda_1^{(k)}\psi_1^{(k)}(\tau_1), \quad 0 \le \tau_1 \le t_0. \qquad (9)$$

Equation (9) can be rewritten as follows:

$$\Omega_k(\tau_i, \tau_j) \triangleq \Omega_{ij}, \quad \tau_i = i\,\Delta, \quad i = 1 \to N = \frac{t_0}{\Delta}$$

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1N} \\ \Omega_{21} & \Omega_{22} & \cdots & \Omega_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{N1} & \Omega_{N2} & \cdots & \Omega_{NN} \end{pmatrix} \begin{pmatrix} \psi_1(1) \\ \psi_1(2) \\ \vdots \\ \psi_1(N) \end{pmatrix} = \lambda_1 \begin{pmatrix} \psi_1(1) \\ \psi_1(2) \\ \vdots \\ \psi_1(N) \end{pmatrix} \qquad (10)$$

$$\Omega \psi_1 = \lambda_1 \psi_1$$

$$\underset{(Nm \times Nm)}{\Omega} > 0 \Rightarrow \lambda_1 > 0.$$

Ideally, at some k we should have

$$\mathbf{f}_k(t) = \sqrt{E}\,\psi_1^{(k)}(t).$$

But in practice they do differ. Define the error signal

$$\mathbf{e}_k(t) = \mathbf{f}_k(t) - \sqrt{E}\,\psi_1^{(k)}(t).$$

Error signal energy is given by

$$\varepsilon_k^2 = \|\mathbf{e}_k(t)\|^2 = E + E - 2\sqrt{E}\underbrace{(\mathbf{f}_k,\, \psi_1^{(k)})}_{\leftrightarrow\, c_1^{(k)}} = 2\sqrt{E}(\sqrt{E} - c_1^{(k)}) \to 0.$$

If $\varepsilon_k \neq 0$, then use the error energy to weight the desired solution $\psi_1^{(k)}(t)$ and provide the correction factor. Thus

$$\mathbf{f}_{k+1}(t) = \mathbf{f}_k(t) + \varepsilon_k \psi_1^{(k)}(t).$$

Equal energy E constraint forces

$$\mathbf{f}_{k+1}(t) = \frac{\mathbf{f}_k(t) + \varepsilon_k \psi_1^{(k)}(t)}{\sqrt{\left(1 + \frac{\varepsilon_k}{\sqrt{E}}\right)^2 - \left(\frac{\varepsilon_k}{\sqrt{E}}\right)^3}}, \quad 0 \le t \le t_0.$$

Conclusions:

- $\mathbf{f}_k(t)$ is seen to converge in all simulated scenarios.
- Shape of $\mathbf{f}_{\text{opt}}(t)$ very much depends on $\mathbf{q}(t)$, $\mathbf{G}_c(\omega)$, $\mathbf{G}_n(\omega)$.
- Unlike the white noise case, the entire target waveform $\mathbf{q}(t)$, $0 \le t \le \infty$, is relevant here to the optimum $\mathbf{f}(t)$ irrespective of the transmit pulse duration t_0.

Receiver Design:

$$\mathbf{H}(\omega) = \mu\{\mathbf{L}'(j\omega)\}^{-1}\mathbf{K}(\omega),$$

where $\mathbf{L}(j\omega)$ is obtained from the spectral factorization

$$\mathbf{L}_f(j\omega)\mathbf{L}^*_f(j\omega) = \sum_{i=1}^{m}\sum_{j=1}^{m} F_i(\omega)F_j^*(\omega)G_{ij}(\omega) + \mathbf{G}_n(\omega)$$

The optimum $\mathbf{f}(t)$ appears ideally to force $\{\mathbf{L}'_f(j\omega)\}^{-1}$ to have flat spectral characteristics. In that case

$$\mathbf{H}(\omega) = \mathbf{K}(\omega) \leftrightarrow \mathbf{g}(t_0 - t)u(t), \qquad \mathbf{h}(t) = \mathbf{g}(t_0 - t)u(t).$$

$$\mathbf{g}(t) \leftrightarrow \{\mathbf{L}'_f(j\omega)\}^{-1}\mathbf{Q}(\omega)\mathbf{F}(\omega) \approx \mathbf{Q}(\omega)\mathbf{F}(\omega),$$

$$\mathbf{g}(t) = \mathbf{s}(t), \quad \mathbf{s}(t) \leftrightarrow \mathbf{Q}(\omega)\mathbf{F}(\omega)$$

$$\mathbf{h}(t) = \mu\, \mathbf{s}(t_0 - t)u(t).$$

Thus optimum $\mathbf{h}(t)$ is also of finite duration.

Point Target: $\mathbf{q}(t) = \delta(t)$

$$\mathbf{h}_{\text{opt}}(t) = \mu\, \mathbf{f}_{\text{opt}}(t_0 - t)u(t).$$

$\mathbf{f}_{\text{opt}}(t)$ in point-target situation with clutter and white noise

is not *any* waveform. Need to solve the nonlinear optimization problem

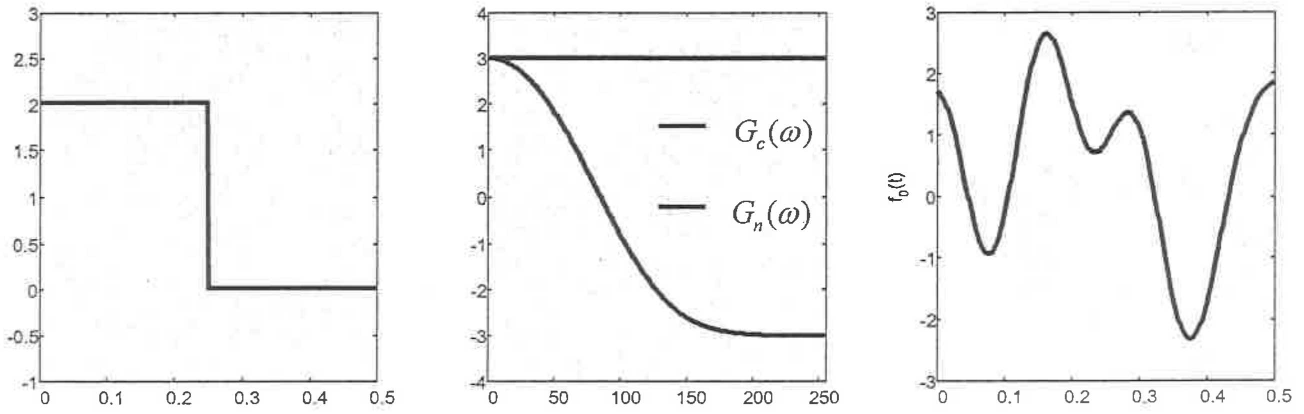
$$\rho(t_0) = \int_0^{t_0} \mathbf{g}'(t)\mathbf{g}(t)dt,$$

where

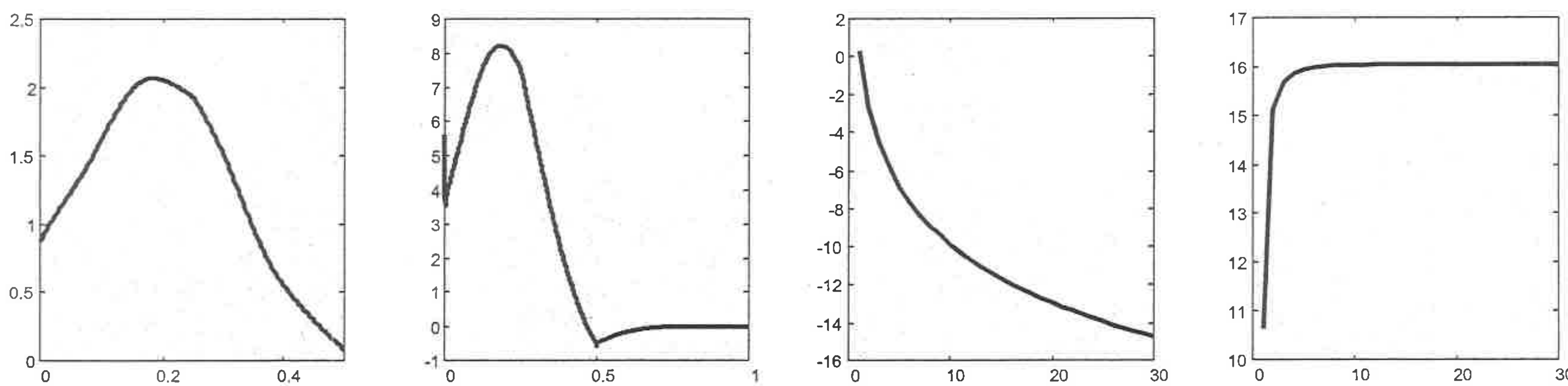
$$\mathbf{g}(t) \leftrightarrow \{\mathbf{L}(j\omega)\}^{-1}\mathbf{F}(\omega),$$

$$\mathbf{L}_f(j\omega)\mathbf{L}'_f(j\omega) = N_0 + \sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{G}_{ij}(\omega)F_i(\omega)F_j^*(\omega).$$

Single Channel Case $q(t)=q_{11}(t),\quad G_c(\omega)=G_{11}^{(11)}(\omega),\quad G_n(\omega)=G_n^{(11)}(\omega)$



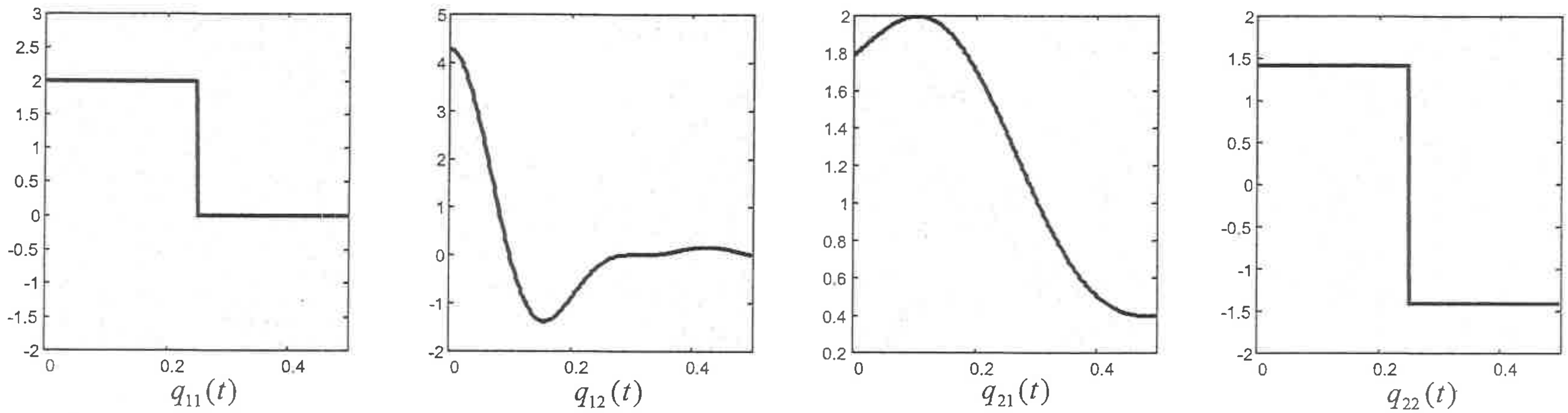
(a) Target $q(t)$ (b) $G_c(\omega), G_n(\omega)$ (c) Initial signal $f_0(t)$



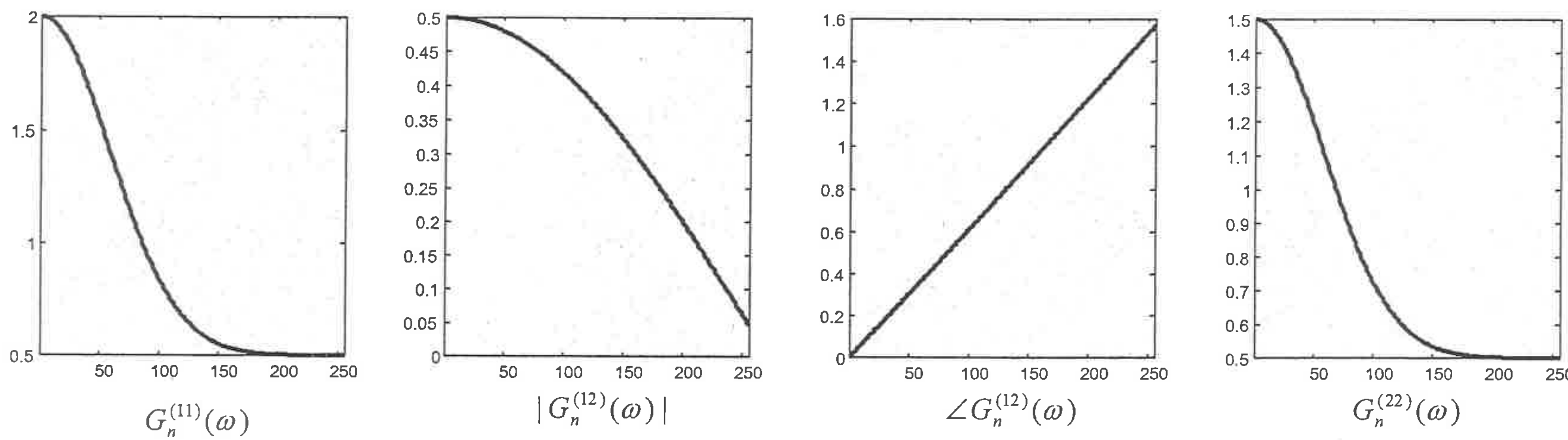
(d) $f_{30}(t)$ (e) Impulse response $h(t)$ (f) ε_k (dB) (f) SINR (dB)

Single Channel Transmitter-Receiver Design

Multichannel Case



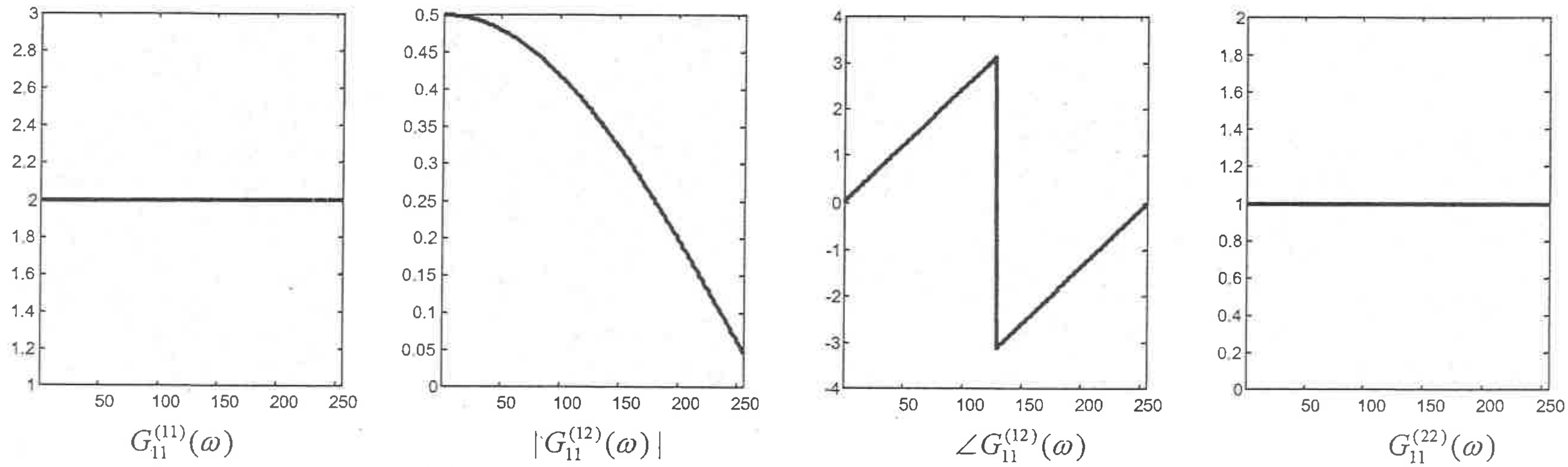
(a) Target impulse response matrix



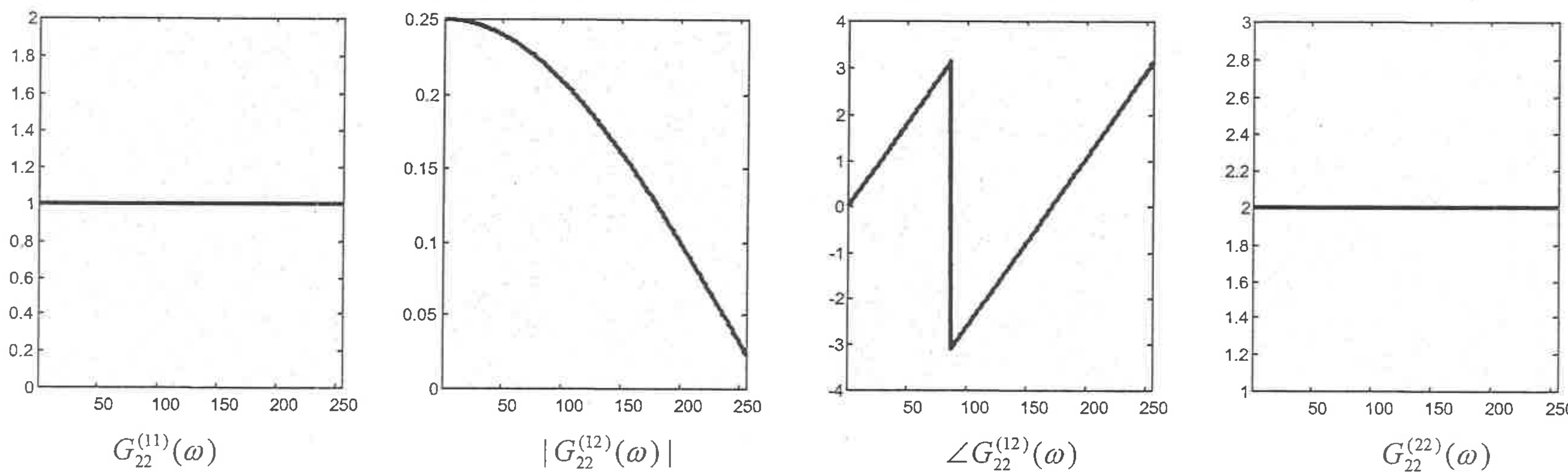
(b) Noise spectral matrix

Clutter Spectra:

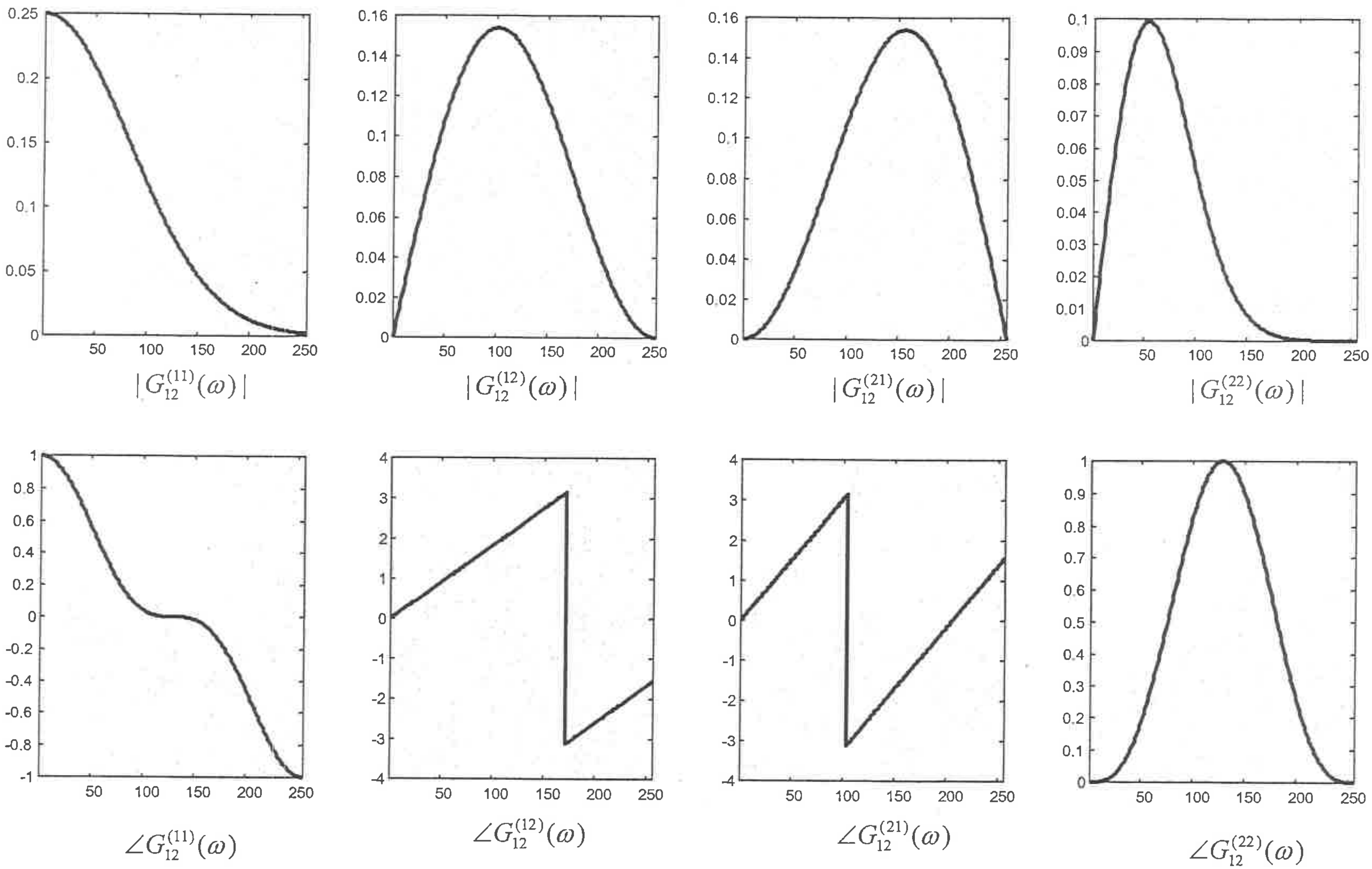
$$\mathbf{G}(\omega)=\begin{bmatrix}\mathbf{G}_{11}(\omega) & \mathbf{G}_{12}(\omega)\\ \mathbf{G}_{21}(\omega) & \mathbf{G}_{22}(\omega)\end{bmatrix},\quad \mathbf{G}_{11}(\omega)=\begin{bmatrix}G_{11}^{(11)}(\omega) & G_{11}^{(12)}(\omega)\\ G_{11}^{(21)}(\omega) & G_{11}^{(22)}(\omega)\end{bmatrix}$$



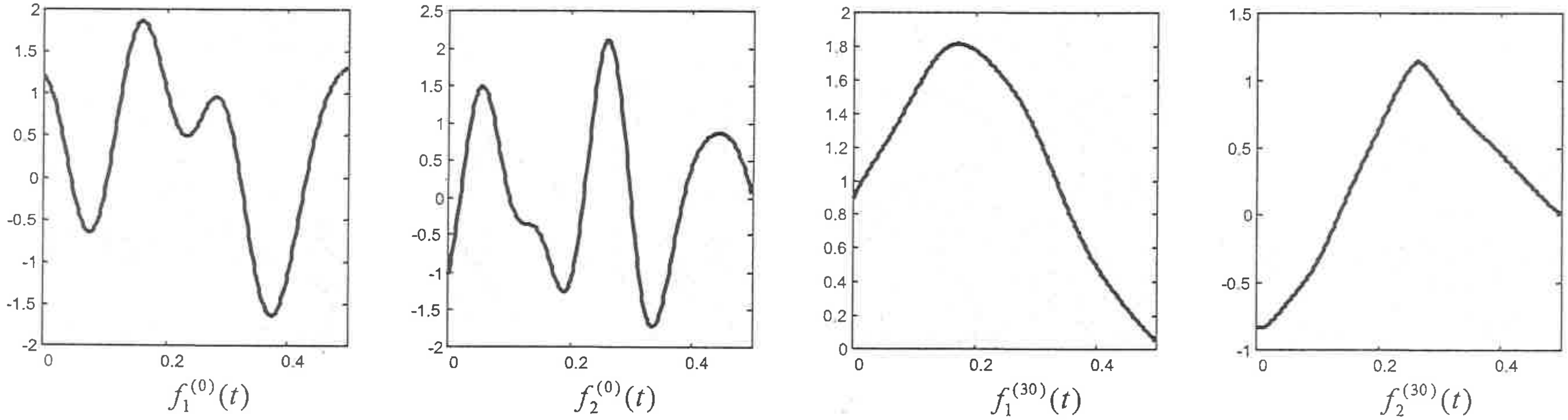
(c) Horizontally polarized clutter spectral matrix



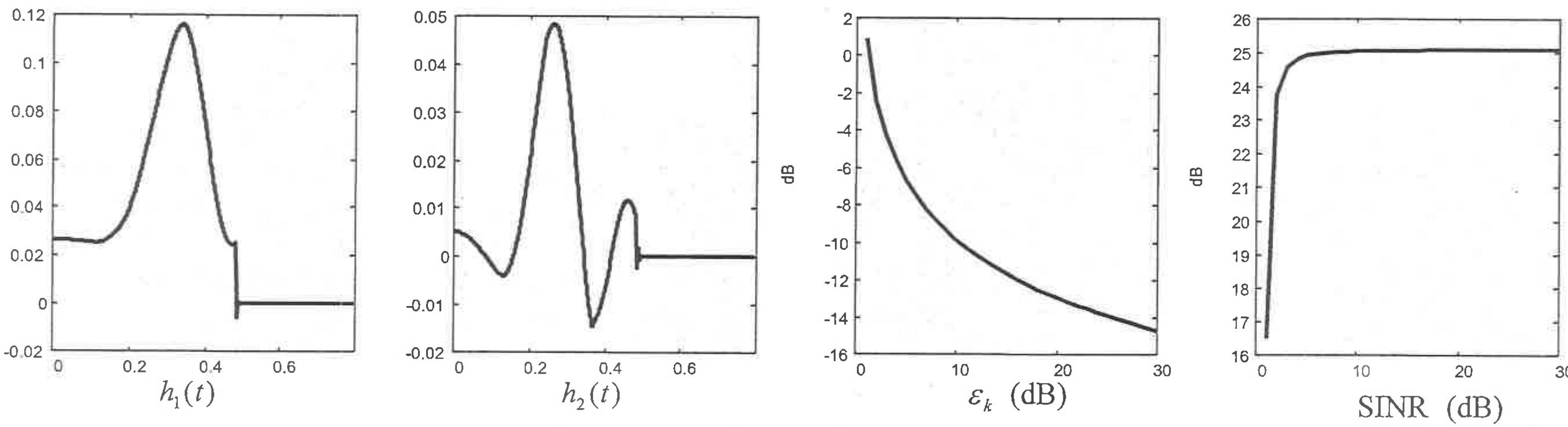
(d) Vertically polarized clutter spectral matrix



(e) Cross polarized clutter spectral matrix



(f) Initial signal vector and optimum signal vector



(g) Optimum matched filter bank and SINR

Remarks

- Single Tx - Rx Case

  $SINR_{max} = 16$ dB.

- Two Tx - Rx Case

  $SINR_{max} = 25$ dB.

In fact, any arbitrary transmit signal vector with both horizontal and vertical polarization achieves at least 16 dB SINR.

Improvement in SINR using Two Tx - Rx = 9 dB.