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Technical Notes and Correspondence

An Invalid Norm Appearing in Control and Estimation

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***Abstract*—This correspondence calls attention to a so-called norm that has appeared and reappeared in the estimation and modern control theory literature. By means of a simple counterexample, the properties that a valid norm should satisfy are shown to be violated by this bogus norm.**

Occasionally analytical misconceptions will arise and even propagate in the literature (e.g., [1] discusses a flaw that was found to be inherent in a very common test for matrix nonnegative definiteness). It is the purpose of this note to call attention to the fairly common error of designating a matrix norm to be the result of finding the minimum of the standard matrix row-sum norm and the standard column-sum norm. This mistake recently occurred in [2, eq. (6)]; however, the oversight did not detract from the validity of the final results. Historically, this error also occurred as far back as 1966 ([3, sec. 3, ch. 2]) in the development of a convergence criterion to signal termination in a computational procedure for numerically evaluating the matrix exponential. Reference [3] served as a model for many packaged computer programs that were subsequently developed by industry and universities for systems analysis and synthesis by modern control techniques. Because of the importance of [3] as an analytic and computational milestone, the effect of this minor misdesignation of a matrix norm in [3] may have had far reaching consequences, which could be deleterious if this so-called norm is used in an application where use of a valid norm is crucial.

The Counterexample

Consider the following two well-known norms of the square matrix A:

$$\|A\|_1 \triangleq \max_j \left\{ \sum_i^n |a_{ij}| \right\} \quad \text{(column-sum norm)} \tag{1}$$

$$\|A\|_2 \triangleq \max_i \left\{ \sum_{j=1} |a_{ij}| \right\} \quad \text{(row-sum norm).} \tag{2}$$

It will now be shown that the following function of A (as used in [2] and [3])

$$\|A\|_3 \triangleq \min\{\|A\|_1, \|A\|_2\} \tag{3}$$

is *not* a norm by demonstrating that it violates the triangle inequality[1] that any valid norm must satisfy ([4, p. 22]). Consider the following square matrices that serve as the counterexample

$$B \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}; \quad C \triangleq \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}; \quad B+C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

Now proceeding with the following intermediate calculations

$$\|B\|_1 = \max\{2,0\} = 2; \quad \|B\|_2 = \max\{1,1\} = 1; \quad \|B\|_3 = \min\{2,1\} = 1 \tag{5}$$

$$\|C\|_1 = \max\{1,1\} = 1; \quad \|C\|_2 = \max\{2,0\} = 2; \quad \|C\|_3 = \min\{1,2\} = 1 \tag{6}$$

$$\|B+C\|_1 = \max\{3,1\} = 3; \quad \|B+C\|_2 = \max\{3,1\} = 3; \quad \|B+C\|_3 = \min\{3,3\} = 3 \tag{7}$$

yields the following result which violates the well-known triangle inequality:

$$3 = \|B+C\|_3 > \|B\|_3 + \|C\|_3 = 2. \tag{8}$$

Even though it is asserted in [3, p. 93] that the function $\|\cdot\|_3$ satisfies the following additional property of norms (that allows a further structure classification as a Banach *algebra* in a *complete* normed space [5, p. 103]):

$$\|BC\|_3 \leq \|B\|_3 \cdot \|C\|_3, \tag{9}$$

the same counterexample that demonstrates that the function $\|\cdot\|_3$ violates the triangle inequality also serves to demonstrate that (9) is also *violated* since

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[1]Since a pseudonorm is only allowed to violate the norm property of being zero if and only if the argument is the null vector, $\|\cdot\|_3$ is not a pseudonorm either.

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$$BC=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{10}$$

$$\|BC\|_1=\max\{2,2\}=2; \quad \|BC\|_2=\max\{2,2\}=2; \quad \|BC\|_3=\min\{2,2\}=2 \tag{11}$$

and, consequently, that

$$2=\|BC\|_3>\|B\|_3\cdot\|C\|_3=1. \tag{12}$$

While, as demonstrated in this correspondence, the minimum of two norms [as in (3)] is, in general, not a norm, the maximum of two norms is always a norm ([4, p. 38]). The usual parallelism between conclusions for the maximum and conclusions for the minimum is absent for the objective of maintaining the properties of a norm.

Acceptable Use as a Convergence Test

A more appropriate norm to use in testing for the convergence of an iterative matrix calculation is the well-known Hilbert norm

$$\|A\|_4=\max_{1<i<n}\left(\lambda_i[A^TA]\right)^{1/2} \tag{13}$$

where $\lambda_i(A^TA)$ represents the ith eigenvalue of A^TA. The norm $\|\cdot\|_4$ is the norm that is naturally *induced* ([6, p. 20]) on the linear operator A from a Euclidean space R^n into itself under the Euclidean norm (i.e., the usual norm) as

$$\|A\|_4=\sup_{\underline{x}\neq 0}\sqrt{(A\underline{x})^T(A\underline{x})}\,/\sqrt{\underline{x}^T\underline{x}}\,. \tag{14}$$

Even though the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are *compatible* ([6]) to the Euclidean norm, the induced norm is naturally tighter ([6, p. 23]) as may be expressed analytically as

$$\|A\|_4<\|A\|_1 \quad \text{for all } A \tag{15}$$

$$\|A\|_4<\|A\|_2 \quad \text{for all } A. \tag{16}$$

From the above two inequalities, the following upper bound may be inferred

$$\|A\|_4<\min\{\|A\|_1,\|A\|_2\} \qquad \text{for all } A \tag{17}$$

(i.e., the minimum of two upper bounds is again an upper bound). Thus it can be guaranteed via (17) that if both $\|A\|_1$ and $\|A\|_2$ are less than some specified criterion, say ε, then $\|A\|_4$ is also less than ε. The computational advantage is in being able to *avoid* the eigenvalue evaluation indicated in (13) and yet be able to guarantee that $\|A\|_4$ is less than ε. The only calculations that are required for this strong conclusion are the relatively simple numerical evaluations of addition and comparison, indicated in (1) and (2), which may conveniently be performed on a computer.

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Output Feedback Decoupling of Multivariable Systems

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Abstract—**Necessary and sufficient conditions for output feedback decoupling of linear time-invariant multivariable systems into single input–multioutput subsystems are derived in frequency domain. The conditions imply a new constructive method for determining a constant gain output feedback matrix and some of its elements can be freely chosen to reassign the closed-loop poles.**

I. Introduction

The decoupling of linear time-invariant multivariable systems using a constant gain output feedback has been discussed in a number of papers. Falb and Wolovich [1] and Howze [2] derived necessary and sufficient conditions in time domain for systems having equal number of inputs and outputs. Wang and Davison [3] and Wolovich [4] investigated the same problem in frequency domain, and more recently, El-Bagoury and Bayoumi [5] employed a frequency domain approach to obtain the desired closed-loop transfer function matrix for systems having unequal number of inputs and outputs.

In this correspondence we will treat the same problem as in [5] in frequency domain and develop necessary and sufficient conditions for decoupling which imply a new flexible method to specify the output feedback gain matrix, where some of its elements can be freely chosen to reassign closed-loop poles of the resulting decoupled systems.

II. Problem

Consider a system characterized by a full-rank, strictly proper transfer function matrix, i.e.,

$$y(s)=T(s)u(s) \tag{1}$$

where $y(s)$ (p-vector) and $u(s)$ (m-vector, $m<p$) are the Laplace transform of the output and the input. We define the linear output feedback law as

$$u(s)=Ky(s)+Gv(s) \tag{2}$$

where K is a constant $m\times p$ matrix, G is a constant $m\times m$ nonsingular matrix, and $v(s)$ is an external m-vector input. Substituting (2) in (1), we have the closed-loop transfer function matrix $T_d(s)$ as

$$T_d(s)=(I_p-T(s)K)^{-1}T(s)G=T(s)(I_m-KT(s))^{-1}G \tag{3}$$

where I_n denotes an $n\times n$ identity matrix.

Then, our objective is to choose K and G so that $T_d(s)$ has nonzero elements only in its main diagonal blocks, where each block is corresponding to a single input—p_i outputs subsystem, $p_i>1$ $(i=1,2,\cdots,m)$, and $\sum_{i=1}^m p_i=p$.

Before proceeding further, we should note some properties of a pseudoinverse matrix which can be found in detail in Rao and Mitra [6].

Lemma 1: The pseudoinverse matrix A^+ of a $p\times m$, full-rank matrix A has the following properties:

a) if $p>m$; $A^+=(A'A)^{-1}A'$, $\quad A^+A=I_m$ (4)

b) if $p<m$; $A^+=A'(AA')^{-1}$, $\quad AA^+=I_p$ (5)

where a prime denotes a transpose of a matrix or a vector.

Lemma 2: If A is a $p\times m$ $(p>m)$ full-rank matrix and B is an $m\times m$ nonsingular matrix, we have

$$(AB)^+=B^{-1}A^+. \tag{6}$$

We now partition $T(s)$ into m blocks, each of which has p_i rows

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