Some Perspectives on Angle-Only Tracking (AOT)*

Thomas H. Kerr, AIAA Senior Member

Abstract

We first provide a detailed up-to-date assessment of the status of Angle-Only Tracking (AOT) technology, a.k.a. bearings-only tracking and discuss general underlying principles for perspective. Once existing problems are exposed and summarized, we then proceed to fix several of them and provide suggested improvements in both the theoretical underpinnings (by addressing both the problem formulation and refinement of regularity conditions relating to "observability" ratings) for at least one particular implementation.

Keywords: Angle-Only Tracking, Bearings-Only Tracking, Kalman filtering, Nonlinear Filtering/Estimation

1 Introduction

Angle-Only Tracking (AOT) technology, a.k.a. bearings-only tracking, is investigated here (since our experience has revealed several unfortunate loose-ends occurring in widely diverse application areas in the open literature on this subject) although there is a fairly wide span of applications waiting to reap the fruits of a consistent theory of AOT from Ballistic Missile Defense [BMD] InfraRed IR target tracking, to passive DIrectional Frequency Analysis and Ranging [DIFAR] (shallow/medium/deep) sonobuoy tracking of ships [1],[2] (as arise in Light Airborne Multipurpose System [LAMPS]), to more recent aero-acoustic target tracking proposed to detect air-breathing cruise missiles, as well as some high-end civilian commercial ultrasonic and IR perimeter monitoring and intruder alert systems.

Once existing problems are properly exposed for a particular AOT formulation (as summarized in Secs. 2, 3.2, 4), we then proceed to fix several aspects of this formulation and to provide suggested improvements (Secs. 2.1, 3.3, and 4) in both the theoretical underpinnings (problem formulation and refinement of regularity conditions related to "observability" ratings) and in recommended implementation. The approach is largely based on advanced Kalman filter and state-space techniques [3]–[5] such as Extended Kalman Filter/Nonlinear Filter Processing [6], [7], Constrained Optimization (viz., [8]–[11]), and conditions for testing Nonlinear Observability [13, p. 415].

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2 An Overview

Our goal is to provide more responsive Kalman-like filters for angle-only tracking (AOT) applications such as for passive sonar/sonobuoy (DIFAR) target tracking as depicted in Fig. 1, passive optical tracking [11], radar target tracking in wideband jammed (range-denied) environment [1] as in Figs. 2 and 3, acoustic tracking of low altitude air breathing cruise missiles, infrared and acoustic tracking (as can arise in maintaining industrial plant security during off-hours).

Our immediate somewhat selfish interest was to investigate this AOT topic further to discern whether enough rigorously substantiated results exist (with a willingness to tolerate mere heuristics, as long as they correspond to good tracking behavior that is repeatable and verifiable) to warrant our introducing AOT as a new capability for our current commercial PC software product: $\mathrm{TK-MIP}^{TM}$. To this end, requisite observability should first be established as the necessary technical or regularity condition that must be satisfied to substantiate that AOT tracking is indeed a well-posed problem that can be expected to yield answers from the efforts of estimation theory and Kalman filter practitioners. The standard straight forward approach to establish observability by just investigating the condition number of an associated Fisher Information Matrix, as is routinely done for linear systems, can't be routinely invoked for the nonlinear AOT scenario. Until just recently, this fundamental question appeared to only be partially answered and even then not very rigorously.

2.1 Observability in AOT formulations Apparently NOW definitively established

We first direct our attention to seeking clear answers to resolve fundamental issues in AOT and seek to obtain a consistent consensus from [16], [31]–[42], [79] on the inherent "observability" underlying the problem as a necessary precursor to being able to solve it satisfactorily. New analytic tools for determining nonlinear controllability (and nonlinear observability as its mathematical dual) have only recently become available [39].

Representative examples are offered in [12] of the importance in control and estimation theory of being able to validly determine numerically whether certain square symmetric matrices are positive definite or semidefinite and to be able to clearly distinguish them from indefinite or negative semidefinite and negative definite matrices. Such a delineation is crucial in KF applications in its own right (within Riccatti eq. calculation of covariances, P, by substantiating proper structure for process and measurement noise covariance intensity matrices and initial condition $P(0)$) and other important applications of such a numerical

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[1] Standard triangulation in a jammed scenario can be accomplished by analyzing the jammed radar's strobe in comparison to that of another cooperating radar [30, p. 1] of known location also in relative proximity (via a synchronized independent comm link and a known baseline between radar locations and by having a capability to identify the jammer in a common reference frame previously agreed upon [such as at a coordinate origin of (N, E, D) located exactly midway on the baseline joining the two radar locations from designated Radar 1 to Radar 2] to indicate target orientation relative to both [or multiple] radars). The position of a single jammer can conceivably be resolved by this joint triangulation technique but the presence of multiple jammers drastically increase the complexity and ambiguity and gives rise to likely ghost targets (that are resolvable by using more sensors with good viewing geometry, i.e., orientation with respect to the targets so proper perspective is availed).

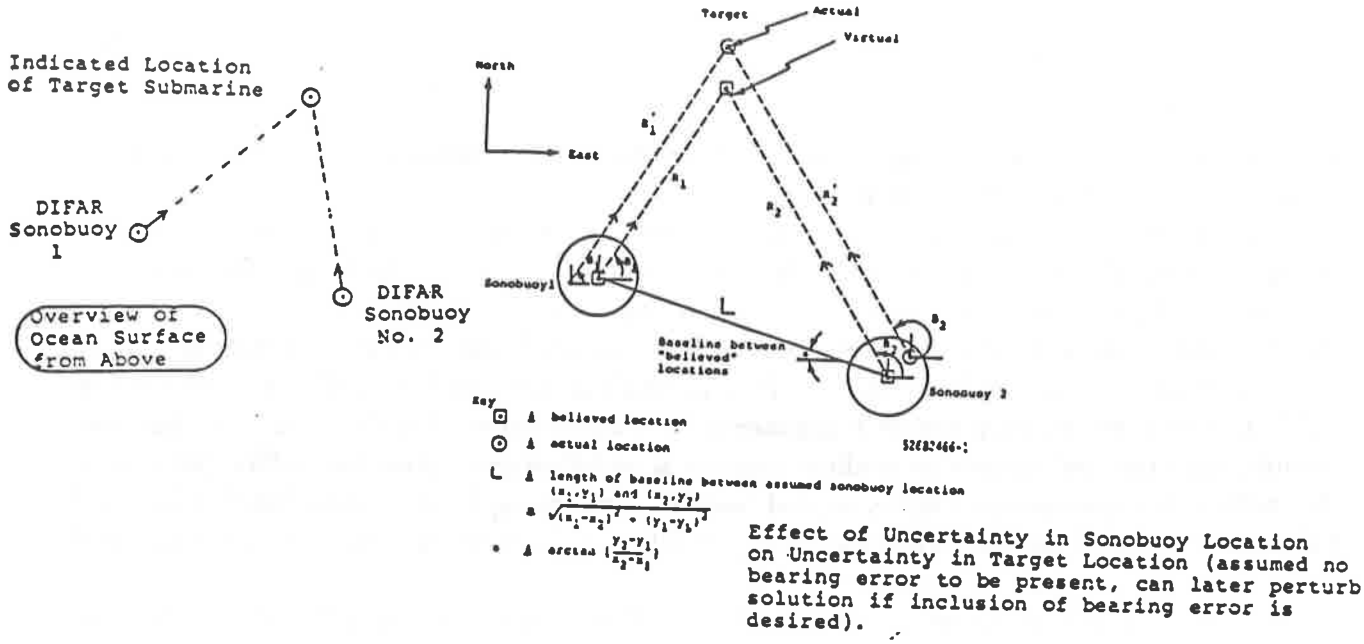

Figure 1: Simplified View of Passive Sonobuoy Target Localization

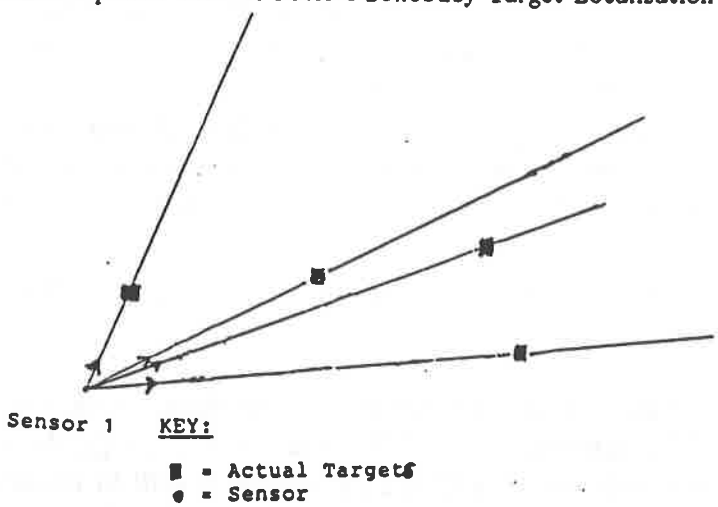

Figure 2: Intersection of lines-of-sight from two coordinated offset sensors reveals actual targets interspersed with fictitious ghost targets (as being initially indistinguishable)

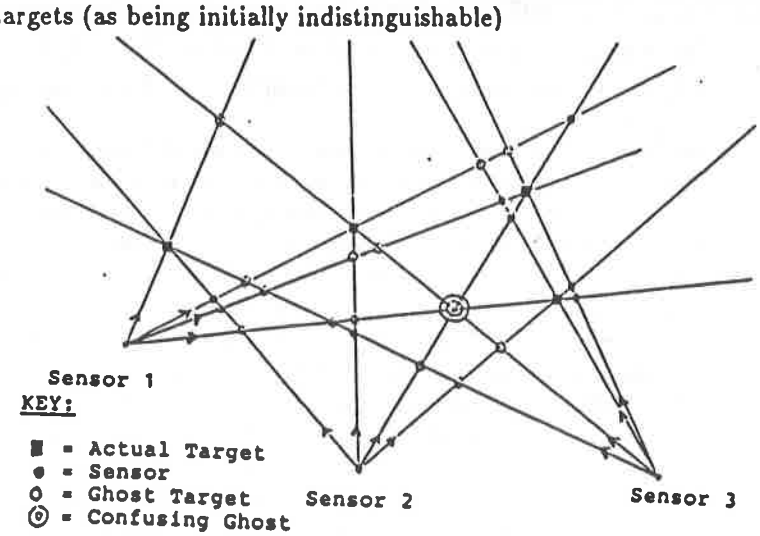

Figure 3: Use of additional sensors helps distinguish most ghost targets

determination arise in confirming *nonlinear observability* via a "strongly positive semidefinite condition" [31], cited in [40]. See Item 7 in Sec. 2.2.

Some prevalent misconceptions on how to test matrices for positive semidefiniteness (both theoretically and computationally) were reviewed in [12]. Although the following so-designated principal minor test for symmetric matrices being that "a matrix is positive semidefinite if and only if its determinant and the determinants of all its principal minors are nonnegative" is a familiar criterion, it is *invalid* (as discussed in [12]). As indicated in [12],[13], there are already ample transparent counterexamples that demonstrate that just considering principal minors to confirm positive semidefiniteness does *not* suffice (although, this author has encountered these cripled tests in performing Independent Verification and Validation IV&V of numerous software Kalman filter tracking implementations for sonar and inertial navigation applications).

Moreover, in recent investigations of observability in 3-dimensional "bearings-only" or "angle-only" applications [41, p. 201], as a precursor to the valid use of an Extended Kalman Filter for target tracking, the prevalent computational test of "nonlinear observability" essentially reduced to a check on matrix positive semidefiniteness, thus providing prior conclusions which may *now* be suspect. Recently all AOT observability questions, worries, and concerns appear to have been completely alleviated and resolved in [79] using the techniques of multilinear algebra (and bang-bang control to prescribe how sensor should move to provide a satisfactory varying perspective to sufficiently delineate target position).

2.2 Persistent problems in some Angle-Only Tracking formulations

Although possessing some aspects that are similar to triangulation and multi-lateration that arise in the field of **Navigation** (c.f., JTIDS RelNav [44], [45]), the theory for performing bearings-only or angle-only tracking (AOT) is evidently still in evolution and is still being worked out [2]! This can be further seen from recent 1990 [14] and 1991 work on this topic [15]–[18] as well as in 1994 [19] and in 1997 [78]. Let's look at this problem once again within the informative context of the historical perspective offered below (for AOT in Ballistic Missile Defense when radars could be range-denied due to interference from enemy RV jammers):

[2]From public database search for recent years, it is apparent that about four Kalman Filter (KF) experts per year are funded to independently look into this angle-only tracking issue. Evidently, it's still not completely resolved (intro of [18] mentions erratic behavior universally observed for Extended Kalman Filters in AOT application and modifications that *may* help overcome this). In 1st paragraph of [23], it is also mentioned that EKF doesn't work satisfactorily in AOT using passive optical (infrared) sensor. Present author's own experience with AOT is that EKF was similarly erratic even though *same* computer code worked well [6] for case when radar range was explicitly available. Also see [46] and enumeration of problems in explicitly determining sufficient Kalman filter "observability" for AOT applications, a type of regularity condition to be discussed further in Sec. 2.1. Similar AOT problems have been observed [2] in passive sonar and sonobuoy target tracking as well. Another critical historical review of AOT developments from a completely different application perspective at Sperry prior to 1969 recently appeared [20] (by no less than one of the first practitioners of age-weighted or fading memory filtering) as this article went to review so author is gratified and relieved to not be out on a limb by himself as the only nay-sayer complaining about the sorry state of existing AOT results in shouting not that "the emperor has no clothes" but that "he is perhaps not yet fully dressed."

1. Jay Sklar established in 1969 that angle-only tracking is possible [30] but in doing so he utilized a *known* launch point and *known* target impact point, as well as *instantaneous knowledge* of missile energy *throughout* the ballistic trajectory (and used the greater data massaging of batch maximum likelihood processing-and NOT a less computationally burdensome Kalman Filter that would at least offer the potential of providing tracking results in real-time);

2. Lou Weiner (then at Teledyne-Brown) performed merely a linear KF *covariance analysis* in 1975-78 and yet strongly concluded that angle-only tracking is practical for Ballistic Missile Defense (BMD). [Without being aware of this precedent at the time, the present author independently verified this step by also using covariance analysis along with the notion of performing target triangulation from the perspective of two cooperating radar, which evidently didn't sufficiently capture the underlying nonlinear aspect of AOT];

3. Bob Miller and C-B Chang used linear Kalman covariance analysis for angle-only tracking in 1977-78 and extracted rules and curves for nomogram evaluation (exoatmospheric case only corresponding to no process noise denoted, in the prevalent filter notation as no Q being present) [24], [25]. They also observed that the assumptions invoked in earlier investigations of information being available on missile trajectory endpoints and continuous access to missile energy via monitoring was not completely realistic for BMD;

4. K-P Dunn and C-B Chang optimized the output of an EKF under a priori likely velocity constraints [26]–[28] in 1979 in performing angle-only tracking based on [27] but Dunn and Chang later returned to using exclusively batch processing instead (an aspect that was undocumented in 1989 and only conveyed verbally);

5. Lincoln Laboratory ostensibly utilizes multi-platform sensors such as Cobra-Eye/Cobra-Ball/OAMP [3] and other ship-based/land-based radar measurement augmentation for guaranteed good viewing geometry (GDOP [4]) through sensor fusion, with hook of having frequently provided augmenting RANGE via any conveniently available radar [5]. When eventually in angle-only mode exclusively (after being well-grounded with some direct range measurements), it still possesses good multi-platform geometry for effective triangulation. Processing is generally not in real-time and the entire data track

[3] Optical Analog Matrix Processing.

[4] Geometric Dilution of Precision relates to the sensitivity of the solution to underlying contributing factors that can be conceptualized geometrically. For example, as in the solution of a system of linear equations, in planar geometry two non-parallel lines intersect in a single point that is the simultaneous solution satisfying both linear equations. If the lines intersect in an angle that is extremely acute, then the location of this solution (intersection point) is very sensitive to uncertainties in the linear coefficients. Likewise for 3- or higher dimensions, where the simultaneous solution of a system of n linear equations is at the common intersection of all the participating planes and is aggravated when some of the participating planes are nearly parallel.

[5] So, strictly speaking, this is **not** exclusively AOT (although advertised as such) because **direct measurement of range is used**. A similar situation (opportunity to cheat) exists in passive sonobuoy DIFAR directional tracking when active sonar LOw Frequency Analysis and Ranging (LOFAR) Range measurements are also interspersed.

file is available to the tracking algorithm since problem definition in this application starts with known launch point and known splash down point as pin-down points at both ends of the target's trajectory in using a non-real-time post-processing Kalman smoother for both forward/backward passes for a least squares type of fit instead of merely relying on a strict KF (that would only process forward in time);

6. The real problem of interest for BMD is to track in REAL-TIME, with target hand-over from coarse-pointing acquisition radar to fine-pointing tracking radar without prior knowledge of exact target destination. Determination of target destination should be an outcome (to facilitate successful interception) [6];

7. Further investigations of the fundamental observability available in angle-only target tracking (AOT) are starting to occur relating to this challenging nonlinear filtering application (see Sec. 2.1) now that adequate tools for doing so are finally just becoming available (e.g., [39] following the lead of [31]–[38]) but no definitive results had been logged for AOT although many showed promise [40], [41], [42] [7] until [79] appeared.

Perceived errors in the approach of Item 4 above are further identified in Sec. 3.2 and a correction is offered in Sec. 3.3.

While multi-target tracking could indeed be interpreted by some to be a Kalman filter adjunct (since it typically uses Kalman filters [for covariance-based adaptive measurement acceptance gates about the predicted state to decide which measurements are associated with existing tracts] or its equivalent asymptotic steady-state versions of α-β-γ trackers), Kalman filtering could just as easily (but more properly) be interpreted to be an adjunct or diminutive special case within the broader topic of multi-target tracking (see [76], [60] for more detail on this last topic in an easy to read form). These multi-target tracking designs take years of planning, coding, and validation (usually by a team such as [21], [22]) unless they are so extremely simplified to be merely planar straight-line constant velocity tracks devoid of maneuvers. The theory of noncooperative multi-target tracking has also evolved over the past 20+ years, as preferred tractable algorithms are still being sought [8] [80]–[84].

[6]In the late 1960's, Dr. D. E. Johansen implemented something less than a real KF for the MIT Haystack radar (according to Dr. J. J. O'Donnell, who investigated this situation ~ 10 years ago). Current philosophy is evidently to use maximum likelihood batch techniques (which are seldom real-time implementations but may now be acceptable due to massive processing power and dedicating hefty MIPS and parallel processing implementation rather than a more sophisticated algorithm design).

[7]From [42, Conclusions], it is claimed that establishing observability of an EFK used for AOT is an easy extension to soon be published by same authors but TeK Associates' subsequent personal correspondence with authors revealed that their originally predicted outcome was not achieved nor is it forthcoming.

[8]The software implementation of the combinatorics and underlying bookkeeping of track files (track initiation, branch on possibility of target splitting into Multiple Independent Retargetable Reentry Vehicles (MIRVs), decoys, and tanks, or bomber launching smart bombs or cruise missiles, incurring track crossing ambiguities, extinctions) and corresponding hypothesis tests and "resource allocation" algorithms for solving the "assignment problem" (e.g., Munkres' Algorithm and Generalized Hungarian Algorithm, Jonker-Volgenant-Castanon (J-V-C) algorithm, 0-1 integer programming solution procedures) is another important aspect that is just as challenging as the fundamental theory if not more so (historically addressed by [59], [60] and in the literature of Operations Research). Most implementations of the track files used to do the bookkeeping that arises within multi-target tracking use linked-lists (standard in LISP, but sometimes even implemented in more readable FORTRAN [and even as a new VBX in Visual Basic called *CANZ* to make it more like the C language] as an extremely useful novelty by Cz Software Corporation).

3 Specific Problems in one Angle-Only Formulation

3.1 Notational convention for modeling Reentry Vehicle target motion

One simple state variable model that *could* be utilized to track Reentry Vehicle (RV) targets by radar or optics describes the position and velocity of the target in terms of the following states

$$x = [R, A, E, \dot{R}, \dot{A}, \dot{E}]^T, \tag{1}$$

where R, A, and E , respectively, denote range, azimuth, and elevation [9]. From [26], the describing differential equations of a nominally non-maneuvering RV, well above earth's atmospheric drag, as seen from a (possibly moving) sensor are:

$$\begin{aligned} \ddot{R} &= R\{\dot{E}^2 + \dot{A}^2 \cos^2(E)\} \\ &\quad - \frac{g_0 R_e^2 \sin(E)}{R_s^2}\left[\frac{R_s^2(R + R_e \sin(E))}{R_T^3 \sin(E)} - 1\right] \end{aligned} \tag{2}$$

$$\ddot{A} = -2\frac{\dot{R}}{R}\dot{A} + 2\dot{A}\dot{E}\tan(E) \tag{3}$$

$$\begin{aligned} \ddot{E} &= -2\frac{\dot{R}}{R}\dot{E} - \frac{\dot{A}^2}{2}\sin(2E) \\ &\quad - \frac{g_0 R_e^2 \cos(E)}{R R_s^2}\left[\left(\frac{R_s}{R_T}\right)^3 - 1\right], \end{aligned} \tag{4}$$

where a dot above a quantity denotes differentiation with respect to time, t, and

$$\begin{aligned} g_0 &\triangleq \text{acceleration of gravity at sea level}, \\ R_e &\triangleq \text{radius of the earth}, \\ R_s &\triangleq \text{vector from center of the earth to the moving sensor}, \\ R_T &\triangleq \sqrt{R^2 + R_s^2 + 2RR_s\sin(E)}. \end{aligned} \tag{5}$$

The above differential equations describe the target as seen from a *moving* sensor (such as an OAMP or spaceborne Strategic Defense Initiative [SDI] midcourse sensor). The effect of gravity enters the above differential equations only in the range and elevation components of the target satellite. Our interest is also in the target as seen from ground-based phased array radar or from ground-based optical sensors. Since the sensor is now assumed to be stationary,

$$R_s \equiv R_e \tag{6}$$

[9] This state selection would also suffice for any exoatmospheric tracking of nonmaneuvering satellites but for endoatmospheric RVs undergoing drag during slow down, an additional 7^{th} state would typically be included in the model so that the filter could also estimate the changing ballistic coefficient as the altitude varied. RV tracking is generally clutter free and free of process noise in the exoatmospheric mid-course regime (phase 2) but noisy upon reentry (phase 3).

in the above and the gravity term in brackets simplifies to no longer include -1 since the now stationary radar or measurement sensor is no longer falling under gravity. In agreement with the corresponding stationary sensor simplification of [26, Eq. 2.6], we now have that

$$\begin{aligned}
\ddot{R} &= R\{\dot{E}^2 + \dot{A}^2 \cos^2(E)\} \\
&\quad - \frac{-g_0 R_e^2 (R + R_e \sin(E))}{(R^2 + R_e^2 + 2RR_e \sin(E))^{3/2}} \\
\ddot{A} &= -2\frac{\dot{R}}{R}\dot{A} + 2\dot{A}\dot{E}\tan(E) \\
\ddot{E} &= -2\frac{\dot{R}}{R}\dot{E} - \frac{\dot{A}^2}{2}\sin(2E) \\
&\quad - \frac{g_0 R_e^3 \cos(E)}{R(R^2 + R_e^2 + 2RR_e \sin(E))^{3/2}}.
\end{aligned} \tag{7}$$

When reexpressed in state variable form, Eq. 7 becomes:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1(x_6^2 + x_5^2 \cos^2(x_3)) \\ -2\frac{x_4}{x_1}x_5 + 2x_5x_6\tan(x_3) \\ -2\frac{x_4}{x_1}x_6 - \frac{x_5^2}{2}\sin(2x_3) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{-g_0 R_e^2 (x_1 + R_e \sin(x_3))}{(x_1^2 + R_e^2 + 2x_1 R_e \sin(x_3))^{3/2}} \\ 0 \\ \frac{-g_0 R_e^3 \cos(x_3)}{x_1 (x_1^2 + R_e^2 + 2x_1 R_e \sin(x_3))^{3/2}} \end{bmatrix} \tag{8}$$

which, upon combining, is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ x_1(x_6^2 + x_5^2 \cos^2(x_3)) - \frac{g_0 R_e^2 (x_1 + R_e \sin(x_3))}{(x_1^2 + R_e^2 + 2x_1 R_e \sin(x_3))^{3/2}} \\ -2\frac{x_4}{x_1}x_5 + 2x_5x_6\tan(x_3) \\ -2\frac{x_4}{x_1}x_6 - \frac{x_5^2}{2}\sin(2x_3) - \frac{g_0 R_e^3 \cos(x_3)}{x_1 (x_1^2 + R_e^2 + 2x_1 R_e \sin(x_3))^{3/2}} \end{bmatrix}. \tag{9}$$

Therefore the describing continuous-time system differential equation above is of the form

$$\dot{x}(t) = f(x(t)) \tag{10}$$

and may be linearized for use within an EKF in the manner detailed in [6, Eqs. 14-34, Fig. 4] (which are converted to an angle-only scenario by reducing or confining the measurement equation to consist of only the corresponding bottom two rows of Eqs. 31, 32 of [6] in directly measuring only noise corrupted azimuth and elevation) in a manner advocated in [26]. As laid out in Secs. 2.1 and 2.2 item 7, this above indicated system and measurement pair needs to be investigated more thoroughly for nonlinear observability.

TeK Associate's direct personal experience in developing EKF software code for this application was that while tracking performance was acceptable for RV tracking when radar range measurements were present [6], performance drastically degraded when range measurements were absent and everything else remained the same. Particularly unsettling was observing immediate divergence of the EKF in AOT mode, even when initial conditions provided to the EKF were exact initial location and velocity of the simulated RV target (an unrealistically benign situation since in actual usage initial conditions would be somewhat fuzzy). While [19] claims to have a remedy against divergence caused by the very first (initial) AOT measurement, [20] complains that this remedy won't necessarily work for similar problems encountered anytime later during the tracking (unless possibly "after each observation step which causes $H\ P = 0$ one should then rotate the uncertainty ellipse about the state estimate so that it is then aligned with the noise's underlying major and minor axes of its uncertainty ellipse").

3.2 Nature of the prior error

The following error was uncovered in the description of an earlier AOT procedure [28, p. 28 in Eq. 3.32], [29, p. 16 in Eq. 2.32]. The nature of the error is clarified in Sec. 4.3 below, and a correction is offered there as well. The gist of the original tracking procedure, which utilizes side information on the maximum and minimum velocities to reasonably expect for the target, does so by using Newton's method to iteratively obtain (at each time step N) the following crucial scalar Lagrange multiplier λ (that arises within the context of an associated constrained optimization, when the constraint is active [10]), according to the following recursion:

$$\lambda_{k+1} = \lambda_k - \frac{f(\lambda_k)}{f'(\lambda_k)}\ , \tag{11}$$

where

$$f(\lambda) = \tilde{x}_N^T P_N^{-1}(P_N^{-1} + \lambda\ S)^{-1}\ S\ (P_N^{-1} + \lambda\ S)^{-1} P_N^{-1} \tilde{x}_N - v_i^2\ , \tag{12}$$

(where $v_i = V_1$ or V_2 depending upon whichever constraint is active at the time); however $f'(\lambda)$ is incorrectly specified in [28], [29] (a situation that we alert the reader to but temporarily ignore for the moment in this section as we continue on with now for purposes of later comparison and to better understand the significance of this historical result but return to properly address in Sec. 4.3) as being

$$f'(\lambda) = -2\ \tilde{x}_N^T P_N^{-1}(P_N^{-1} + \lambda\ S)^{-1}\ S\ (P_N^{-1} + \lambda\ S)^{-1} P_N^{-1} \tilde{x}_N\ . \tag{13}$$

It is further suggested in [28], [29] that Eq. 11 above be iterated until

$$|\lambda_{k+1} - \lambda_k| < \epsilon \tag{14}$$

as a stopping criterion, where typically ϵ is a very small number in practice (such as, say, $\epsilon = 10^{-6}$ for example).

Further examination of Eqs. 11 to 13, reveals the following structure (if Eq. 13 were in fact correct):

$$NUMERATOR = -\frac{1}{2}\ DENOMINATOR - v_i^2 \tag{15}$$

[10] To force calculated Reentry Vehicles (RV's) estimated velocities to be strictly between two *a priori* imposed (and specified) lower and upper bounds, V_1 and V_2, respectively.

thus causing Eq. 11 to simplify as

$$\begin{aligned}\lambda_{k+1} &= \lambda_k - \frac{NUMERATOR}{DENOMINATOR} \\ &= \lambda_k - \frac{-\frac{1}{2}DENOMINATOR - v_i^2}{DENOMINATOR} \\ &= \lambda_k + \tfrac{1}{2} + \frac{v_i^2}{DENOMINATOR}\,,\end{aligned} \tag{16}$$

where the $DENOMINATOR$ (corresponding to the RHS of Eq. 13) is clearly negative when the scalar λ is positive (as it must be to satisfy the Kuhn-Tucker conditions of constrained optimization), and P_N is positive definite (by computation from the matrix Ricatti equation within the EKF), and S is positive semidefinite (by construction). By examining Eqs. 14 and 16 together, it can now be seen immediately that the iterations of Eq. 11 will terminate only when the following condition is satisfied:

$$\frac{1}{2} + \frac{v_i^2}{-2\,\tilde{x}_N^T W \tilde{x}_N} < \epsilon\,, \tag{17}$$

where

$$W \triangleq P_N^{-1}(P_N^{-1} + \lambda\, S)^{-1}\, S\, (P_N^{-1} + \lambda\, S)^{-1} P_N^{-1}\,. \tag{18}$$

Notice that as λ approaches 0 from above as a limit, then $W \rightarrow S$ and, consequently, $\tilde{x}_N^T S \tilde{x}_N$ is apparently nonnegative. The second term on the left in Eq. 17 is apparently always negative (or more exactly nonpositive) and the iteration equation of Eq. 11 will *terminate* (signifying convergence) if the principal denominator term $\tilde{x}_N^T W \tilde{x}_N$ is large enough so that Eq. 17 is strictly satisfied. Going further, the iteration of Eq. 11 (as seen by examining Eq. 16) will ideally reach a steady-state answer when $\lambda_{k+1} = \lambda_k$ for every succeeding value of k, which corresponds to:

$$0 = \lambda_{k+1} - \lambda_k = \frac{1}{2} + \frac{v_i^2}{DENOMINATOR}\,. \tag{19}$$

or (from Eq. 17)

$$0 = \frac{1}{2} + \frac{v_i^2}{-2\,\tilde{x}_N^T W \tilde{x}_N}\,, \tag{20}$$

which reduces to

$$v_i^2 = \tilde{x}_N^T W \tilde{x}_N\,. \tag{21}$$

The question logically arises whether any nonnegative λ can be found that satisfies Eq. 21, expanded out here to reveal its inner structure to be:

$$v_i^2 = \tilde{x}_N^T P_N^{-1}(P_N^{-1} + \lambda\, S)^{-1}\, S\, (P_N^{-1} + \lambda\, S)^{-1} P_N^{-1} \tilde{x}_N\,. \tag{22}$$

If Eq. 22 cannot be satisfied for any non-negative λ, then Eqs. 11 and 16 will never converge. Evidence suggests that Eq. 22 can be satisfied for some λ but the utility of this conclusion (based on an erroneous Eq. 13 is now being challenged below). This is enough discussion on the original formulation from [28], [29] since our claim or assertion here is that it is in error. Substantiation of this claim and a corrected derivation is provided next.

3.3 Our proposed correction and revised derivation/mechanization

Before becoming immersed in the derivation, for proper insight let us first recall a simple result as the tool that we will need. From Eq. 12 above, certain matrices were encountered of the form $(P_N^{-1} + \lambda S)^{-1}$. It is important to know how to *correctly* differentiate this term with respect to the scalar λ. To this end, observe that (using a standard contrivance similar to that used to derive the expression for the derivative with respect to time of the *inverse* of a time-varying matrix):

$$(P_N^{-1} + \lambda\, S)^{-1}(P_N^{-1} + \lambda\, S) = I\,, \tag{23}$$

so that applying the chain-rule in taking the derivative with respect to λ yields

$$\begin{aligned}&[\tfrac{\partial}{\partial\lambda}(P_N^{-1} + \lambda\, S)^{-1}](P_N^{-1} + \lambda\, S) + (P_N^{-1} + \lambda\, S)^{-1}[\tfrac{\partial}{\partial\lambda}(P_N^{-1} + \lambda\, S)]\\ &= \tfrac{\partial}{\partial\lambda} I \equiv 0\,,\end{aligned} \tag{24}$$

which upon rearranging and simplifying becomes

$$[\frac{\partial}{\partial\lambda}(P_N^{-1} + \lambda\, S)^{-1}] = -(P_N^{-1} + \lambda\, S)^{-1} S (P_N^{-1} + \lambda\, S)^{-1}\,. \tag{25}$$

This simple result (somewhat familiar in form due to invoking the standard contrivance mentioned above) is key to the correction in what follows.

Returning now to the derivation of the constrained optimization (as laid out in [28, pp. 22-28], [29, pp. 15-17]), consider the minimization over $\hat{x}$ (in a six-dimensional real Euclidean space [11]) of the scalar cost function:

$$J = (\hat{x} - \tilde{x}_N)^T P_N^{-1}(\hat{x} - \tilde{x}_N)\,, \tag{26}$$

(with $\tilde{x}_N$ being the estimate already availed from the update step of the EKF at time N, P_N being the covariance of estimation error already availed from the update step of the EKF at time N) subject to the following constraint on allowable estimated velocities:

$$V_1^2 \le \hat{x}^T S \hat{x} \le V_2^2\,, \tag{27}$$

where V_1 and V_2 are specified numbers and

$$S = \begin{bmatrix} 0_{3\times3} & \vdots & 0_{3\times3} \\ \cdots & \cdot & \cdots \\ 0_{3\times3} & \vdots & I_{3\times3} \end{bmatrix}, \tag{28}$$

and $I_{3\times3}$ is an identity matrix. The vector variable $\hat{x}$ that is optimized (as the solution to the above problem described by Eqs. 26, 27) is to be the constrained estimator $\hat{x}_N$ that is sought (as some additional processing of the output of the EKF). While it is conceivable that this constrained estimate could in turn be feedback to the EKF to be utilized in calculating the

[11]Recall that the underlying state variable model utilized within the methodology of [25]-[27], is in Earth Centered Earth Fixed (ECEF) coordinates and has the first three states being those of position and the next (and last) three states being those of velocity.

EKF propagate step for time $N+1$, it does not appear to be used this way in the approach of [28, Fig. 3.4], [29, p. 7, Fig. 2.2] but is instead used as a stand-alone add-on.

Least it be missed, the evident interpretation of Eq. 27 (upon expanding it out) is:

$$V_1^2 \quad \leq \hat{x}_4^2 + \hat{x}_5^2 + \hat{x}_6^2 \equiv v_x^2 + v_y^2 + v_z^2 \equiv |\text{velocity}|^2 \quad \leq V_2^2 \,. \tag{29}$$

For an intuitive $\mathcal{R}^n \times \mathcal{R}^1$ geometric interpretation (not present in [28] or [29] which does offer a less informative $\mathcal{R}^n$-plane compressed projection), the optimization problem stated here in Eqs. 26 to 28 is as depicted in Fig. 4 (corresponding to Eq. 26 alone for ease in visualization) as in Fig. 5 (corresponding to Eq. 27 alone for ease in visualization) and as summarized jointly in Fig. 6 (for the joint constrained optimization of Eqs. 26 and 27 together).

Further pursuing the explicit algebraic equations that describe the constrained optimization that is to be performed, if upon performing just the minimization of Eq. 26 (without considering the constraint of Eq. 27), the global answer will either also satisfy Eq. 27 (in which case the constrained answer will be identical to the unconstrained answer and the constraints of Eq. 27 haven't inhibited this answer so the constraints would be interpreted as being inactive) or it won't. If the global optimization of exclusively Eq. 26 fails to satisfy Eq. 27, it is either because

$$|\text{velocity}|^2 < V_1^2 \,, \tag{30}$$

or because

$$V_2^2 < |\text{velocity}|^2 \,, \tag{31}$$

in which case either the portion of the constraint:

$$|\text{velocity}|^2 = V_1^2 \,, \tag{32}$$

or the portion of the constraint:

$$|\text{velocity}|^2 = V_2^2 \,, \tag{33}$$

is *active*, respectively. While the intermediate considerations just conveyed were for insight, the entire constrained optimization can be handled in one fell swoop, as embodied by finding the saddle point in the following Lagrangian (as theoretically justified by invoking Kuhn-Tucker's theorem):

$$\mathcal{L}(\hat{x}, \lambda) = (\hat{x} - \tilde{x}_N)^T P_N^{-1} (\hat{x} - \tilde{x}_N) + \lambda[\hat{x}^T S \hat{x} - v_i^2] \,. \tag{34}$$

The reason why a general v_i^2 appears in Eq. 34 rather than V_1 or V_2 explicitly is that it is physically impossible for both extremes of this velocity constraint to be active at the same time (i.e., the unconstrained velocity can't be both below the lower reasonable RV velocity bound V_1 and above the higher reasonable velocity bound V_2, simultaneously) so only one extreme is depicted as being active at a time (time=N) in Eq. 34 above [12].

Proceeding to find the saddle points of Eq. 34 by differentiating Eq. 34 separately, first with respect to the unknown $\hat{x}$, and then with respect to the unknown λ (then set both equal to zero to find the critical points) yielding:

$$[\frac{\partial}{\partial \hat{x}} \mathcal{L}(\hat{x}, \lambda)] = 2P_N^{-1}(\hat{x} - \tilde{x}_N) + \lambda[2S\hat{x}] \equiv 0 \,, \tag{35}$$

[12] Compare with [28, Eq. 3.25], where λ is initially incorrectly depicted as being a vector rather than a scalar Lagrange multiplier.

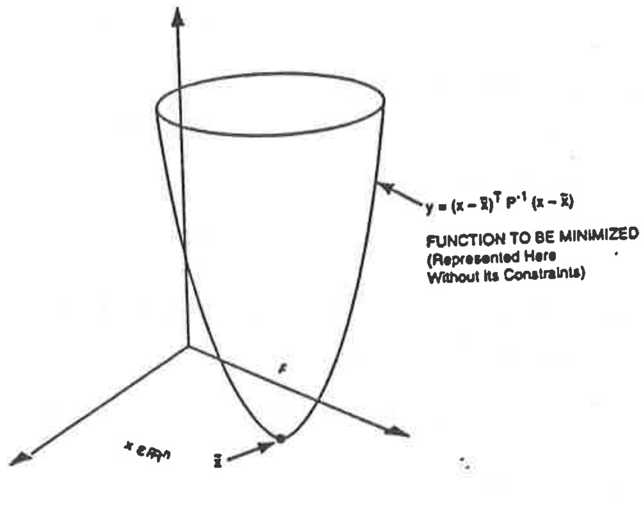

Figure 4: Global Optimization, as called for in Eq. 26.

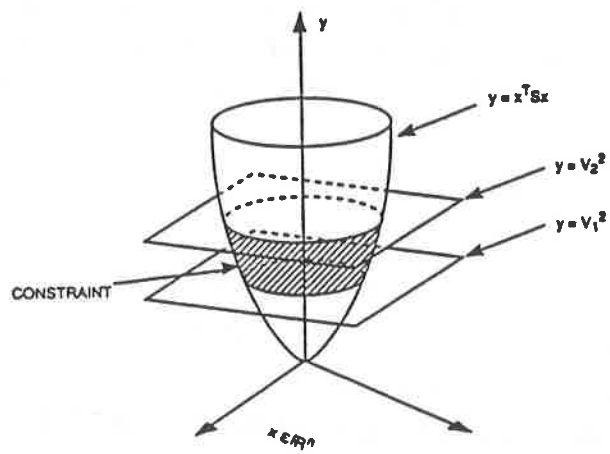

Figure 5: Constraints to be Satisfied, as called for in Eq. 27.

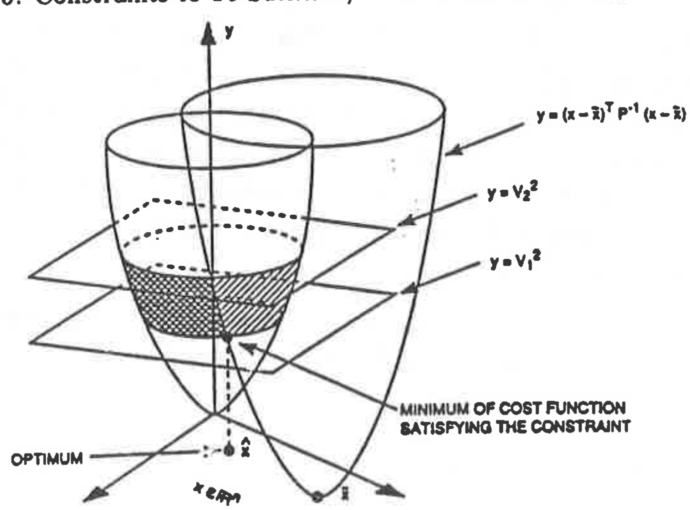

Figure 6: Constrained Optimization, as called for in Eqs. 26 and 27, together.

and

$$[\frac{\partial}{\partial \lambda}\mathcal{L}(\hat{x},\lambda)] = [\hat{x}^T S \hat{x} - v_i^2] \equiv 0 \, . \tag{36}$$

(and, in typical fashion, the differentiation of the Lagrangian above with respect to λ just results in the equation of the constraint again, as an internal cross-check).

Upon solving Eq. 35 for $\hat{x}$ yields [13]

$$\hat{x} = (P_N^{-1} + \lambda \, S)^{-1} \, P_N^{-1} \, \tilde{x}_N \, . \tag{37}$$

When this representation of the solution is substituted back into the constraint equation of Eq. 36, the new representation of the constraint is

$$\tilde{x}^T P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1} \tilde{x} = v_i^2 \, . \tag{38}$$

To satisfy both Eqs. 37 and 38 as the saddle point, we must have a solution for the scalar λ in the above. An approach that comes to mind is to use Newton's method as discussed in Eqs. 11 and 12 from Sec. 3.2 (and Eq. 11 correctly corresponds to our Eq. 38 here that is to be satisfied). However, notice that while $f'(\lambda)$ needs to be correctly specified in order to validly apply Newton's method, the expression of Eq. 13 is *not correct*. The correct expression is *now offered here*. Using the lemma of Eq. 25, it is now easily seen that $f'(\lambda)$ should be

$$\begin{aligned}
& f'(\lambda) \triangleq \frac{\partial}{\partial \lambda}[f(\lambda)] \\
& = \tilde{x}^T P_N^{-1} \left[\frac{\partial}{\partial \lambda}(P_N^{-1} + \lambda \, S)^{-1}\right] S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1} \tilde{x} \\
& + \tilde{x}^T P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S \left[\frac{\partial}{\partial \lambda}(P_N^{-1} + \lambda \, S)^{-1}\right] S P_N^{-1} \tilde{x} \\
& = -\tilde{x}^T P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1} \tilde{x} \\
& \tilde{x}^T P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1} \tilde{x} \\
& = -2\tilde{x}^T P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1} \tilde{x}
\end{aligned} \tag{39}$$

The most that can be rigorously said about the $DENOMINATOR$ of Eq. 11 as it now correctly occurs in Eq. 39 is that it is negative semidefinite (and that hopefully $\tilde{x}$ is not in the null space of $P_N^{-1}(P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} S (P_N^{-1} + \lambda \, S)^{-1} P_N^{-1}$ otherwise the $DENOMINATOR$ will blow up. While this author had a previous fulfilling experience in analyzing an algorithm in [8] for calculating a scalar Lagrange multiplier of somewhat similar structure, the structure here of Eq. 11 in conjunction with that of Eqs. 12 and 39 precludes further direct conclusions by being less tractable for analysis than the *harder* optimization problem of [8] that had its own separate equation for λ (that was demonstrated in [8] to be a geometrically convergent contraction mapping, when used as an iteration equation) and so didn't have to resort to using Newton's method.

Using the relationship of Eq. 37, it is easy to deduce by post-multiplying by its transpose and taking expectations throughout that the associated covariance or uncertainty in constrained estimation is

$$\tilde{P} = (P_N^{-1} + \lambda \, S)^{-1} \, P_N^{-1} \, (P_N^{-1} + \lambda \, S)^{-1} \, . \tag{40}$$

[13] Here we also follow a prevalent slight abuse of notation (as also used in [28], [29]) rather than use the symbol $\hat{x}$ to denote the optimized value only after Eq. 35 has been solved (as should be its rigorous usage), instead here and in [28], [29] the symbol $\hat{x}$ is used throughout the entire development/derivation.

This is a new result here that was not in [28], [29] but is the answer to a natural question of how good is this constrained estimate? The composite calculations needed to compute the covariance result of Eq. 40 are already present so it is no significant additional computational burden to provide this also as the usual gauge that should always accompany a state estimate.

Repeating the important message from [28], [29] here for emphasis, although the constraint that is imposed forces velocity estimates from the imposed (designated) "reasonable range", it may be seen from Eq. 37 that in so doing *all the states are altered* to result in the final constrained estimate and that, in particular, *the position estimates will also be modified as a consequence.*

4 Suggestions to Improve Radar Target Tracking of RV's

4.1 An approach for Testing & Validating (Debugging) nonlinear filtering software

While Gaussians are present throughout standard KF application analysis, they are generally absent in nonlinear filtering aplications. In contrast, Gaussian inputs into a linear system yield Gaussian outputs even if the linear system is time-varying. A Gaussian is completely characterized by its mean (conditional estimate: $\hat{x}$) and variance P (as it evolves from the associated Riccati equation). The presence of Gaussian process and measurement noises are usually argued via the Central Limit Theorem that sums of multitudinous independent entities add up and go to Gaussian in distribution. However, even the presence of Gaussian noises in nonlinear systems (described by nonlinear ordinary differential equations) yields outputs that are unlikely to be Gaussian and perhaps not even unimodal. Since non-Gaussian outputs and estimates are encountered, they need more than just the mean and estimate for a full characterization. In general, all the moments and cross-moments (or, equivalently, cummulants or semi-invariants) must be specified for a complete characterization of non-Gaussians so the problem is generally infinite dimensional. Differential equations can be specified for the time evolution of all higher moments but they are, in general, coupled with the time evolution of even higher order moments [88, p. 7], [89].

An exact finite-dimensional optimal nonlinear filtering test case of the type discovered by Benes and extended by Daum [14] [50] (with a recent rigorous update in [51]) may suffice for IV&V in the same manner as [52] (for linear Kalman filters) by providing collaborative comparison of outputs to verify performance of a general EKF implementation (instantiated with the same test case) if both implementations agree (sufficiently) for this simple test. This proposed manner of use for EKF software verification would be in keeping with the overall software test philosophy being espoused in [5].

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[14] Another nonlinear filtering example with a finite-dimensional implementation, not covered within the situations addressed in [48] and [50], is for scalar system $\dot{x} = f(x) + g\,\xi(t)$, where $E[\xi(t)\xi(s)] = q\,\delta(t-s)$ and $f(x) = -\frac{x}{1+x^2}$. The verifiable asymptotic solution in the limit as t goes to infinity of the associated Fokker-Planck or forward Chapman-Kolmogorov equation (defined in [49, pp. 126-130]) is $p(x,t|z,s) = \frac{c}{(1+x^2)^{c_2}}$, where $c_2 = \frac{1}{g^2 q}$ and c is the normalization constant for this pdf.

Table 1: Stages and Confirmation Tests for Phased EKF Development

STAGE	SOFTWARE UTILIZED	COVARIANCE ANALYSIS	FILTER PORTION	LINEARIZATION	RV TRAJECTORY GENERATOR	PURPOSE
1	RVTRIANG	Covariance only.	None	About true trajectory.	Straight line.	Establish benchmark for later comparisons.
2	Modified RVTRIANG (copy 1)	Introduce EKF Covariance mechanization (except linearized about true states).	Introduce EKF Filter portion (except linearized about true states). Print/plot states.	Same as above (so not a true EKF yet).	Same as above. Need to print/plot states.	See if covariances are similar to above case. See if filter estimates follow true trajectory (maybe with lag).
3	Modified RVTRIANG (copy 2)	Introduce linearization about estimates (so now a true EKF).	Introduce linearization about estimates (so now a true EKF).	Introduce linearization about estimates (so now a true EKF).	Same as above.	See how closely EKF estimates follow true trajectory.
4	Modified RVTRIANG (copy 3)	Same as above.	Same as above.	Same as above.	Introduce nonlinear equations for conic RV trajectory.	See how closely EKF estimates follow true trajectory.
5	Modified RVTRIANG (copy 4)	Same as above.	Same as above.	Introduce relinearization.	Same as above.	See improvement in how closely EKF estimates follow true trajectory.

Elaborating further on the above proposed use of Benes/Daum filters only for IV&V of EKF's or Gaussian Second-order filters, it is reminded that nonlinear filtering usually has an infinite dimensional solution in general (that is not practicable in general) and use of EKF's or Gaussian second order filters are finite dimensional approximations that are *sometimes* adequate. The Benes/Daum filters are extremely special cases of nonlinear filter problems that don't correspond to RV tracking at all but offer a finite dimensional implementation (without being approximate) to certain "toy problems". One way to constructively use these known results is to IV&V a general EKF implementation to see if it yields similar answers for the same toy problems. Then apply EKF implementation to actual problem at hand after it passes this initial sanity check (to reveal any problems lurking in the EKF implementation).

Actual experience in developing an EKF for angle-only Reentry Vehicle (RV) tracking via jammed (range-denied) radar using triangulation (RVTRIANG), as modified from an earlier EKF for tracking RV's via unjammed radar, convinced this author that such goals are best carried out in specific well thought out stages. Examples are, first, for a constant gravity, then for inverse-squared gravity. First, for a non-rotating earth, then for a rotating earth. More detail on this aspect is provided in [6, footnotes 5 and 8]. A representative plan for EKF development that the author has previously successfully adhered to for this type of endeavor is depicted in Table 1.

4.2 A caution against using Age-Weighted Filtering

Mechanizing as a *limited memory filter* or as an *age-weighted filter* (which uses a weighting to emphasize more heavily the measurements obtained in the recent past [53]–[57]) to avoid the following problem, otherwise encountered, of the filter becoming *oblivious* to later measurements because its bandwidth has already closed-down following receipt of the totality of earlier measurements which it will otherwise weigh equally with the current measurements thus dismissing the importance of the relatively small number of recent measurements as being unable to in any way alter its already perceived trend [15].

4.3 A caution regarding use of GLR

While Generalized Likelihood Ratios (GLR) (where maximum likelihood estimates of unknown parameters are utilized within the likelihood ratios in lieu of the parameters being unknown) are presented and developed by Davenport and Root [63], Root went further [64] to investigate applicability of GLR techniques in the radar detection problem of resolving closely spaced targets in either a background of known arbitrary correlated Gaussian noise or Gaussian white noise. However, Root [64] obtained explicit criteria that could be applied to indicate conditions under which one could reasonably expect to NOT be able to resolve two known signals (of unknown amplitudes and parameters) and additionally pointed out a difficulty of using GLR for this purpose.

Selin [65] found that some of the unknown parameters (such as unknown relative carrier phase) must also be estimated in order to maximize the *a posteriori* probability in the estimation of two similar signals in white Gaussian noise. Selin further identified four standard caveats [66, p. 106] associated with the use of a maximum likelihood estimate of the unknown parameters in a likelihood ratio (as utilized in GLR).

McAulay and Denlinger [70] advocated use of GLR in conjunction with a Kalman filter in decision-directed adaptive control applications. Finally, Stuller [67] defined an M-ary GLR test that ostensibly overcame Root's original objections [64] to GLR for this type of application. ([67] also provides a limited history of GLR developments for radar, excepting no mention of [70], which possibly eluded him.

The use of GLR for failure detection was pioneered by Willsky and Jones [71] using an identical GLR formulation as presented by McAulay and Denlinger [70]. While both Willsky and McAulay claim optimality of the GLR they never explicitly specify a criteria by which it may be judged optimal nor do they supply a proof or reference where such a claim is demonstrated (specifically, [70] references the proof to be in a english translation of an identified German textbook but diligent follow-up revealed no such substantiation).

On [68, p. 92], attention is called to the fact that GLR is not a Uniformly Most Powerful (UMP) test, while [68, p. 96] offers recognition that cases exist where use of GLR can give bad results. That a maximum likelihood estimate (MLE) is not necessarily statistically consistent in general is explicitly demonstrated in a counterexample in [69, p. 146]

GLR is again being advocated for use in radar applications [73], [74], [75] but appear to ignore the historical objections for use of GLR in these types of applications as well as the explicit counterexamples in [72, 968 ff, App. A, pp. 973-974] that have never been refuted. An alternative to the use of GLR where there are critical unknown parameters is the use Expectation-Maximization, known as the E-M algorithm.

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[15] While enthusiastically and extensively endorsed in the past as a technique to be used to effect improvements in filter performance, use of age-weighted or fading-memory filtering is now known to not always improve the situation and to frequently actually aggravate it [58].

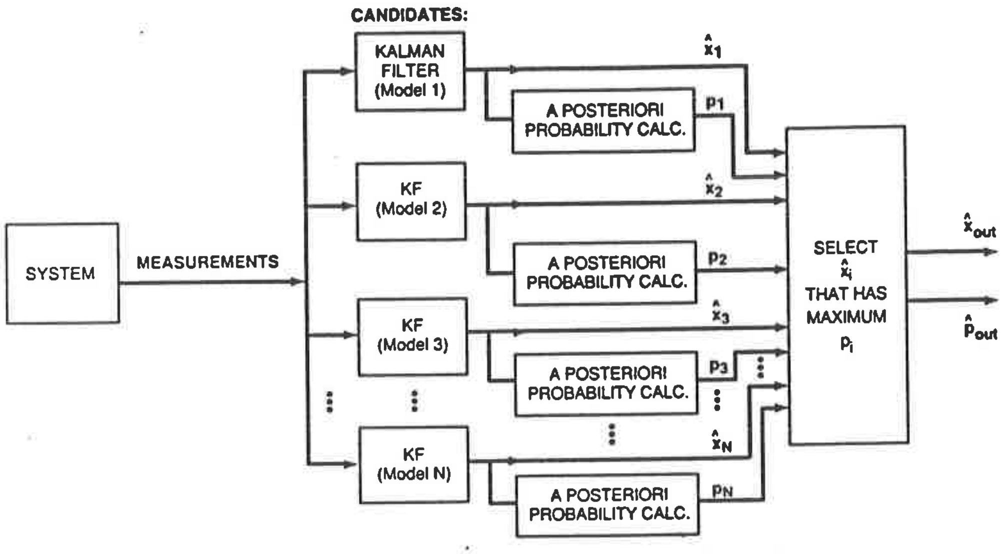

Figure 7: Multiple Model of Magill (MMM): N alternative filters, each with its distinctly different system models, vying to match the true (unknown) system as it progresses through its likely operating regimes (characterized a priori by analysts), with associated on-line computation of probabilities of each being correct so that a tally is available to decide which one (choice of a "winner" varying with time) offers the best match

4.4 Potential Benefits of a Bank-of-Kalman-Filters Approach

We feel compelled to suggest possible parallel implementation of the "Bank-of-Kalman-Filters" approach [85] (where each filter has a different underlying system model matched or representing a different hypothesized underlying situation) with global probability assessments of each filter possibly coinciding exactly with the true situation (currently prevailing and from which the only measurements are availed throughout) being automatically calculated on-line as an integral part of this methodology, which is totally rigorous only for linear systems. (As originally conceived in 1965 by Magill, popularized by Demitri Laniotis as "partitioned filters", but only relatively recently pursued for actual use by R. Grover Brown, Peter Maybeck, Yaakov Bar-Shalom, and Wang Tang (ARINC) within the last 15 years in IR, GPS, Radar, and multi-target sonar and radar applications with significant extensions being provided in the last six years by Y. Bar-Shalom, H. Blom, and X.-R. Li [86], [87].)

Regarding the utility of using a "bank-of-Kalman-filters" approach [16] for reentry-phase RV target tracking please consider the following possibilities:

- use of alternative atmospheric models upon reentry, where arguments arise as to exactly

[16] A detail is that originally Magill, D. Laniotis, John Deyst [consulting at TASC prior to his Draper affiliation], Charles Brown (then at TASC) used manimization to pick highest probability and choose just a single estimate as the winner, while present day implementations blend the estimates as weighted by corresponding probabilities. Mike Athans et al was the first to blend outputs like this in 1977 Oct. issue of *IEEE Trans. on Automatic Control* on "Fly-by-Wire Control of the F-8 NASA Test Aircraft" but they acknowledged that it was heuristic, at least it was in their LQG feedback control formulation.

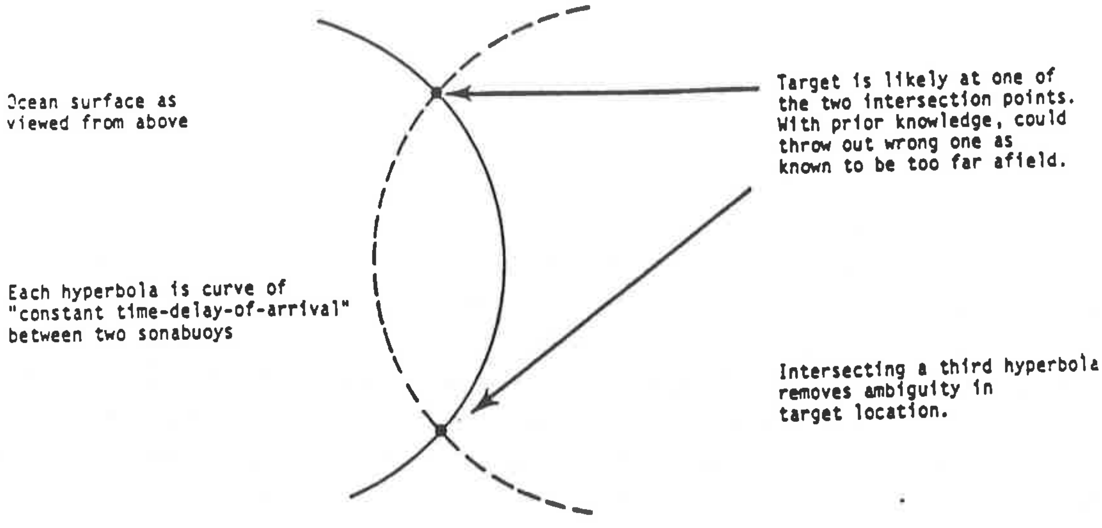

Figure 8: Principles of Sonobuoy Target Localization inferred from Intersections of Hyperbolas

what altitude it kicks in.

- alternative RV masses hypothesized (quantized over the finite possibilities aided by prior intelligence gathering to elucidate candidates).
- quantized on possibilities on spin modulation speed (if any) as elucidated by prior intelligence gathering.
- quantized over likely reentry angles (which affects drag and lift). Different countries of origin use different conventions on reentry angle (but may alter at the last minute to reap the element of surprise just like in the electronic Intelligence [ELINT] game of Electronic Warefare [EW]).

A "bank-of-Kalman-filters" is also being used in some simplified approximate multitarget tracking methodologies such as the Joint Probabilistic Data Association (JPDA) scheme advertised by Y. Bar-Shalom[17] as being a lesser computational burden than full Multi-Hypothesis Test (MHT) approaches of Fred Daum (Raytheon) or Sam Blackman (Hughes).

[17] Yaakov Bar-Shalom and Hank Blom also use a generalization of MMM (denoted as IMM) and have a nice description of the accompanying probability calculations of IMM, which in turn, determines which running filter model most closely corresponds to actual measurements received. Bar-Shalom and X.-R. Li have recently extended this structure to automatically close down on the number of model filters to avoid an excess of candidates (that would otherwise drain computer resources and water down tracking performance as well).

5 Improvements in Post-Coherence Target Localization in Sonobuoy Tracking

A lucrative approach, not yet addressed for the sonobuoy application to our knowledge, is the benefit of coordinating target tracking filters on separate cooperating surveillance platforms using different sonobuoy mixes as the following decentralized approach to data fusion to enhance detection and estimation/tracking of enemy targets. Just as in the Navy Tactical Data System (NTDS), separate platforms can utilize the GRID LOCK mode to cross-corroborate radar images from the differing perspectives of separate platforms and to use the known position of "friendlies" appearing on the radar screen as an anchor to sift out false alarms for a better assessment of the correct number of "unfriendlies" present by appropriately combining some of the separate blips into one where there is only one hostile target in that vicinity.

Conceptually, the process of post-coherence function target localization is depicted in Fig. 7 with the intersection of two (or more) hyperbolas (corresponding to two distinct sensor pair correlations providing constant differences in delay time-of-arrival [61, p. 1500]). This process is analogous to the operation of a hyperbolic LORAN-C radio navigation system (in the standard hyperbolic mode as distinguished from the rho-rho mode).

However, like LORAN [62], post-coherence target localization can also suffer from the effects of bad geometry corresponding to what may be incurred by LORAN users in encountering bad Geometric Dilution of Precision (GDOP) if they are in an unfavorable location with respect to the LORAN transmitter sites within radio reception range. Moreover, unlike what is simplistically depicted in Fig. 7, the problem is NOT strictly planar for sonobuoy applications because of (1) possible ray bending in the rather nonlinear acoustic medium due to the thermal gradient and (2) the fact that when depth is properly considered, the curve of constant delay-time-of-arrival is actually a slightly contorted/distorted hyperbolic surface (of one sheet). Even with these minor complications, several hyperbolic surfaces may be intersected (by intelligent selection of sonobuoy pairs to participate in the processing [18]) to yield a unique solution to target localization.

The least sensitivity to error in ultimate target localization solution is obtained when the hyperbolic lines or surfaces intersect at almost 90° angles. This favorable situation occurs when the baselines drawn between the locations of the sonobuoys utilized are themselves approximately orthogonal. Such considerations can be utilized as the theoretical basis of a proposed "Executive Sonobuoy Selection Algorithm" to automatically select (as an operator aid) the subset of available sonobuoys to participate as part of the target localization process. Such a subset selection would enhance the GDOP associated with calculating the solution as the target localization yet avoids less productive sonobuoy processing (as a reasonable way to conserve scarce resources for this application).

Please notice that the computations constituting the proposed Executive Sonobuoy Selection Algorithm are all simple and straightforward and may be accomplished from a simplified planar viewpoint in selecting (i.e., recommending to the human operator) sonobuoy partici-

[18] A historical constraint in LAMPS is that processing power is finite and that modest limits are imposed on the subset of sonobuoys that may be included in the target localization computations from the many more sonobuoys already seeded and available for processing.

pation or preferential ordering based on a criteria of having most nearly orthogonal baselines (or ordering based on having the largest angle less than 90° between baselines). Baselines are to be erected conceptually between all candidate sonobuoys and simple easy inner product calculations suffice to establish angles and hence the sorting or ordering criteria for selecting the sonobuoys to participate in the further processing for target localization.

6 Conclusion

We breifly surveyed the breath of applications awaiting a consistent AOT tracking methodology. Kalman filters for linear, possibly time-varying, totally controllable and observable systems are *robust* with respect to bad initial guesses on initial conditions by converging exponentially to true values despite bad initial guesses that are far off the mark. As a nonlinear filtering problem, AOT is particularly challenging by exhibiting great sensitivity to initial conditions or initial estimates. Moreover, despite claims to the contrary, historical observability investigations for AOT applications were flawed until just recently. Futhermore, even AOT formulations that persisted over decades had serious flaws in the underlying theory. One apparently flawed approach was examined here and corrected. A few other loose-ends in related areas were corrected here as well and cautions were issued where warranted to encourage more practical implementations in sonobuoy and RV tracking.

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