and observable for each k, $k = 1,2,\ldots$, and if $A_k \to A$, $\Gamma_k \to \Gamma$, $C_k \to C$ for some constant matrices A, Γ, and C as $k \to \infty$, with the steady-state system (A, Γ, C) being both completely controllable and observable, then the covariance of the error term ϵ_{k_0+l} given in (7) tends to zero exponentially as $l \to \infty$.

This theorem can be easily verified by elementary limit arguments, but is very tedious in the mathematical expressions. Hence, we omit the details here.

Finally, we remark that in case more than one single bit of data are missing, there will be infinitely many cases to be considered and it is impossible to discuss them very precisely. Besides, it is most unlikely that we can do (sub)optimal filtering under the condition that too much data information have been missed. Hence, we do not go any further beyond the scope of our investigation of the problem in this short note.

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Fallacies in Computational Testing of Matrix Positive Definiteness/Semidefiniteness

Rounding out three prior critical comments on pervasive misstatements of tests for matrix positive definiteness/semidefiniteness, this work reveals yet other aspects where common fallacies have arisen in this critical area and this assertion is demonstated via a concrete yet transparently simple *original* counterexample. A corrected theoretical formulation as well as a practical computational implementation for a proper version of a test of this property is also offered here.

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I. INTRODUCTION

Some prevalent misconceptions on how to test matrices for positive semidefiniteness (both theoretically and computationally) were reviewed in [1]. Although the following so-designated principal minor test for symmetric matrices being that "a matrix is positive semidefinite if and only if its determinant and the determinants of all its principal minors are nonnegative" is a familiar criterion, it is *invalid* (as discussed in [1]). As indicated in [1], there are already ample transparent counterexamples (as in [2], [3]) that demonstrate that just considering principal minors to confirm positive semidefiniteness does *not* suffice. Several engineering applications are described in [1, 1st paragraph of column 2, p. 504]. Moreover, in recent investigations of observability in 3-dimensional "bearings-only" or "angle-only" applications [4, p. 201], as a precursor to the valid use of an extended Kalman filter for target tracking, the prevalent computational test of "nonlinear observability" essentially reduced to a check on matrix positive semidefiniteness, thus providing prior conclusions which may *now* be suspect.

An area of caution is that several prominent textbooks have stumbled into the same analytic pitfall of interpreting the test for positive semidefiniteness in too strong an analogy with the valid principal minor test for positive definiteness (also referred to as "Sylvester's criterion"). Other recent, otherwise mathematically rigorous, textbooks [5, Theorem 1.4.2, p. 14; 6, Note 11.2, p. 468] have also fallen into this same trap. For positive *semidefiniteness*, all $2^n - 1$ possible subminors ([3]) and not just the leading principal minors need to be considered in order to have a valid test that is both necessary *and sufficient.* In [1] for balance, several textbooks were identified that have a correct statement of the test for positive semidefiniteness. However, as discussed in [1], for practical problems of realistically higher dimensions, the evaluation of multiple minors or determinants as offered in these correct textbooks is not a computationally efficient approach for determining whether a matrix is positive definite, negative definite, semidefinite, or indefinite. A preferred approach is to make use of the singular value decomposition (SVD) in making such a determination, as explained in [1]. Since SVD has historically been observed to be the only computationally reliable method for establishing the rank of a matrix, the refinement of SVD, known as Aasen's method, which exploits underlying symmetry of the matrices (and only requires on the order of $n^3/6$ operations, where n is the dimension of the square matrix under test) is the currently available test of least computational burden for numerically performing these types of definiteness determinations, as discussed in [7]. However, a direction is offered in Section VII for a faster test than this if it can be recast as a systolic array, as is currently being pursued by others.

In [8], simple counterexamples revealed that a recently offered partitioned test for demonstrating the positive semidefiniteness of a matrix (with the potential of being applied stagewise to the higher dimensional matrices encountered in industrial applications) is flawed. A proper version of such a test was uncovered, as historically developed by others within the totally different application context of matrix spectral factorization, but which is also valid in the simpler case considered in [8], where the matrices of interest have constant numerical entries. All these prior aspects have already been raised and resolved in [1–3, 7, 8] but were reviewed here to support comparisons with the extension treated in Section VII.

More fundamental aspects of symmetric positive *definite* matrices (encountered in applications involving covariance matrices) are examined here in Section II and prevalent "reasonableness tests," recounted in Sections III and IV, are revealed in Sections V, VI to be fallacious via an *original* transparently simple counterexample. Again, a corrected theoretical formulation as well as a practical computational implementation for a proper version of a test of this property is provided in Section VII.

II. ELEMENTARY STRUCTURAL CONSIDERATIONS

Two well-known structural properties of an arbitrary covariance matrix (for a real random vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]^{\mathrm{T}}$) of the form:

$$P = E[(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^{\mathrm{T}}] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2n} \\ p_{31} & p_{32} & p_{33} & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & \cdots & p_{nn} \end{bmatrix} \quad (1)$$

(where, in the above symmetric matrix, $p_{ij} = p_{ji}$ and $E[\cdot]$ is the expectation operator) are reviewed here prior to discussing previously used but less transparent computational "reasonableness tests" in Sections III and IV. This review of elementary covariance structure is pursued here first in order to facilitate later demonstration in Sections V and VI that these historical "reasonableness tests" are fallacious by construction of a viable counterexample.

As is fairly well known, any proper covariance matrix P can be represented in the following form ([9, p. 91, eq. (3–76)]):

$$P = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \cdots & \cdots & \rho_{1n}\sigma_1\sigma_n \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \cdots & \cdots & \rho_{2n}\sigma_2\sigma_n \\ \rho_{31}\sigma_3\sigma_1 & \rho_{32}\sigma_3\sigma_2 & \sigma_3^2 & \rho_{34}\sigma_3\sigma_4 & \cdots & \rho_{3n}\sigma_3\sigma_n \\ \vdots & \vdots & \rho_{43}\sigma_4\sigma_3 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \rho_{n1}\sigma_n\sigma_1 & \rho_{n2}\sigma_n\sigma_2 & \rho_{n3}\sigma_n\sigma_3 & \cdots & \cdots & \sigma_n^2 \end{bmatrix} \quad (2)$$

where the σ_i's in the above are the standard deviations and where the correlation coefficients satisfy

$$\rho_{ij} = \rho_{ji}$$

since the above covariance matrix is always symmetric. Moreover, as is true of all correlation coefficients (defined in terms of the matrix entries of (1)):

$$\rho_{ik} \triangleq \frac{p_{ik}}{\sqrt{p_{ii}p_{kk}}} \quad (4)$$

and their magnitudes should theoretically be constrained to

$$-1 \le \rho_{ik} \le 1 \qquad \text{for all } i,k \quad (5)$$

(as can be rigorously verified by applying the Cauchy-Schwarz inequality to the underlying scalar components of the vector random variables that correspond to the entries of P in (1) yielding

$$E[(x_i - \bar{x}_i)(x_k - \bar{x}_k)] \le \sqrt{E[x_i^2 - \bar{x}_i^2]}\sqrt{E[x_k^2 - \bar{x}_k^2]} \quad (6)$$

which simplifies to be

$$p_{ik} \le \sqrt{p_{ii}}\sqrt{p_{kk}}. \quad (7)$$

Rearranging (7) and using the definition of (4) yields (5).

Another standard structural observation is that a covariance matrix P, such as that depicted in (2), can be further decomposed as

$$P = \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \sigma_3 & & \\ & 0 & & \ddots & \\ & & & & \sigma_n \end{bmatrix} \times \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \rho_{23} & \cdots & \cdots & \rho_{2n} \\ \rho_{31} & \rho_{32} & 1 & \rho_{34} & \cdots & \rho_{3n} \\ \vdots & \vdots & \rho_{43} & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \rho_{n3} & \cdots & \cdots & 1 \end{bmatrix} \times \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \sigma_3 & & \\ & 0 & & \ddots & \\ & & & & \sigma_n \end{bmatrix} \tag{8}$$

which is of the form of

$$P = A^{\mathrm{T}} B A \tag{9}$$

where

$$A \text{ is of full rank}$$

as long as the standard deviations are nondegenerate as

$$\sigma_i > 0 \qquad \text{for each} \quad i = 1, \ldots, n. \tag{10}$$

It is well known that *if* a matrix B is positive definite, as denoted by

$$B > 0$$

then postmultiplying B by a full rank matrix A and premultiplying by the transpose of the same matrix A preserves positive definiteness as [10, p. 13][1]

$$A^{\mathrm{T}} B A > 0. \tag{11}$$

Therefore, when all standard deviations are given to be strictly greater than zero as in (10), then P is positive definite if and only if B is positive definite. It then suffices to only check B for positive definiteness in order to infer whether P, as related to B via (9), is positive definite. Not only positive definiteness, but negative definiteness, semidefiniteness, and indefiniteness are preserved by transformations of the form of (9). This property is utilized in constructing the counterexample of Section V.

[1]For other insights into the algebra of validly manipulating positive definite matrices to preserve positive definiteness, please see [11, Appendix].

III. AN HISTORICALLY USED TEST OF POSITIVE DEFINITENESS

For two decades up to the present, I have frequently encountered the following test advertised as a computational sanity check or "reasonableness test" that has been used on the covariance of estimation error that arises in real-time Kalman filter applications (ranging from submarine navigation systems ([11], [12]), to sonobouy tracking, to military aircraft navigation ([13]) and target tracking). The test essentially consists of first checking for the most obvious or flagrant violations by confirming that no diagonal term is nonpositive, then subsequently considering all possible 2×2 submatrices of the original $n \times n$ matrix P of (1) and testing that all entries of these (and the original matrix) satisfy

$$|p_{ik}|^2 < p_{ii} p_{kk}. \tag{12}$$

Only if the above condition is satisfied for all $i, k = 1, \ldots, n$ is the computed matrix P deemed acceptable. Usually, this is the only "test of goodness" for the P matrix in these real-time applications and the test is advertised as ostensibly being a test of positive definiteness (or, at worst, a test of positive semidefiniteness) having the major attraction of not requiring an exorbitant number of operations in the central processing unit (CPU) time-stingy real-time environment.

The act of checking the inequality of (12) can be recast as merely calculating the determinant of the following 2×2 submatrix and testing it for nonnegativity:

$$\det \begin{bmatrix} p_{ii} & p_{ik} \\ p_{ik} & p_{kk} \end{bmatrix} = p_{ii} p_{kk} - p_{ik}^2 > 0 \tag{13}$$

[as further recomputed for *all* possible 2×2 submatrices]. The precise number of these 2×2 minors D_2 within a general $n \times n$ matrix, as n things (or entries) taken two at a time, is provided by the binomial coefficient being $(n/2) = n!/(n-2)!2! = (n^2 - n)/2$. Using these combinatorics, offered here for analyzing the version of the test in (13), the number of operations can be seen to be 2 multiplies, a squaring (or another multiply), a subtraction, and a comparison for each of the $(n^2 - n)/2$ matrix evaluations per an application of this test which, along with checking the main diagonal terms for nonpositive entries as n comparisons, yields a total of $4((n^2 - n)/2) + n = 2n^2 - n$ operations. For the version of this test in (12), the combinatorics-based operations count is $3((n^2 - n)/2) + n = \frac{3}{2}n^2 - \frac{1}{2}n$ since one less operation is used in (12).

The above test described in this section has become a folk-theorem since it is pervasive, yet no proof has ever been supplied of its validity. Conversely, no one has challenged its veracity or even *questioned* its validity in the open literature or otherwise until now, as offered in Section V.

IV. AN HISTORICALLY USED METHOD OF ENFORCING WELL-CONDITIONING FOR COMPUTED POSITIVE DEFINITE MATRICES

It is fairly well known that the round-off and trunction errors encountered in the implementation of the otherwise ideal mathematical operations of addition, subtraction, division, and multiplication by a particular machine can frequently accrue to such an extent that computed matrices that would theoretically be positive definite can in fact become so ill-conditioned that main diagonal terms may actually become zero or perhaps even go negative (both occurrences being immediate anathema or contradictory to having a theoretically positive definite matrix).

Such ill-conditioning has a particularly disasterous effect in implementations of a Kalman filter covariance that is recursively propagated in time and which consequently adversely affects the calculation of the filter gain, which further adversely affects the quality of the filter estimates and can even compromise the stability of the filter proper (which relies on an associated Lyapunov function, constructed from the inverse of the ideal positive definite covariance of estimation error as the theoretical substantiation of the stability of the filter, as discussed in [14, Sect. 4.1, 4.2, Appendices A.1, A.2] by causing a type of divergence [15]). In an attempt to avoid or, more accurately stated, to seek to benignly control possible ill-conditioning of the computed covariance matrix, one prevalent technique has emerged and is referred to here as covariance check (COVCHK). The COVCHK approach (usually included within a single COVCHK software routine) is to first test for zero elements on the principal diagonal. If any of these diagonal elements are zero (to within the precision of the machine) as a disasterous event that cannot be conveniently side-stepped, then the entire covariance is directly reinitialized via a PINIT procedure; otherwise, positivity of the main diagonal terms of the $n \times n$ matrix are routinely further enforced at each computed time-step by making the following assignments:

$$p_{ii}^{*} \leftarrow |p_{ii}| \qquad \text{for} \quad i = 1 \text{ to } n \tag{14}$$

(which can be seen to force all main diagonal terms to be nonnegative despite what was originally computationally obtained). Further, all the off-diagonal elements of P are revised by forming an auxiliary matrix P^{*} in its stead according to the following rule:

$$p_{ik}^{*} \leftarrow \frac{p_{ik}^{2}}{p_{ii}p_{kk}} \qquad \text{for} \quad i,k = 1 \text{ to } n \tag{15}$$

then subsequently checking the auxiliary matrix P^{*} to confirm that each

$$p_{ik}^{*} \leq 1 \tag{16}$$

otherwise, if the condition of (16) is not satisfied, then an even more benign off-diagonal term is created of the form:

$$p_{ik}^{*} \leftarrow \frac{p_{ik}}{\sqrt{p_{ik}^{*}} + 0.001} \qquad \text{for} \quad i = 1 \text{ to } n \tag{17}$$

and the entire resulting matrix is checked (ostensibly for positive definiteness) using the test described in Section III as (12). The mathematical significance of using the above matrix alterations in an attempt to maintain positive definiteness of the covariance matrices being computationally encountered is described in Section VI.

V. COUNTEREXAMPLE TO TEST OF SECTION III

A crucial observation concerning the condition of (12) is that by dividing both sides by the positive quantities $p_{ii}p_{kk}$ yields a result that corresponds to the condition of (5) (or (15) and (16)). The multiple calculations of the form of (12) (or (13)) amount to no more than merely confirming that the underlying cross-correlation coefficients have magnitudes less than 1. That it does no more than just this in seeking to offer protection as an advertised "reasonableness test" is demonstrated by the following counterexample.

Consider the following specific 3×3 symmetric matrix:

$$B_1 = \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} & 1 \end{bmatrix}. \tag{18}$$

This matrix will now be checked for *positive definiteness* using the principal minor test ([8, pp. 381–382]); also known as Sylvester's criterion ([9, 13]) consisting of the following three steps:

Step 1:

$$b_{11} = 1 > 0 \tag{19}$$

Step 2:

$$\det\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \det\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} = \frac{3}{4} > 0 \tag{20}$$

Step 3:

$$\det[B_1] = -\frac{7}{16} < 0. \tag{21}$$

If Step 1 through Step 3 above yielded *all* positive numbers, then B_1 would have been demonstrated to be positive definite. However, since this was *not* the case, it is evident that B_1 is not positive definite. Futher evidence is that when the above B_1 is used as the weighting matrix in a scalar quadratic form $f(x) = x^{\mathrm{T}}B_1x$, the resulting specific evaluations can be both positive and negative depending on the choice of the vector since $x_1 = [1,1,1]^T$ yields $f(x_1) = \frac{9}{2}$ while $x_2 = [1,1,-1]^T$ yields $f(x_2) = -\frac{1}{2}$. If B_1 were indeed positive definite, all evaluations of the quadratic form would be positive for $x \neq [0,0,0]^T$. The above two alternate characterizations used to dispute the positive definiteness of B_1 was easier to invoke than actually calculating and exhibiting its eigenvalues since that would necessitate solving a general cubic equation that is avoided here by the approach used.

Consider now the 3×3 case of matrix P_1 constructed according to (8) using an A_1 having nonzero standard deviations:

$$0 < \sigma_i \qquad \text{for each} \quad i = 1,2,3 \tag{22}$$

then P as defined by (8) (of the form of (9)) using the above B_1 of (18) would be

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}^T \begin{bmatrix} 1 & -\frac{1}{2} & \frac{3}{4} \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & 4 \\ 3 & 4 & 16 \end{bmatrix} \tag{23}$$

which, by the discussion at the end of Section II, has the same definiteness as that possessed by B_1!

Now in performing the test of Section III on the matrix P_1 of (23), notice that the principal diagonal terms satisfy the first requirement of being positive. Notice from the form of the matrix B_1 in (18), that all the correlation coefficients are exposed in the off-diagonal terms and that each has a magnitude less than 1 to satisfy the condition of (5) (or (15) and (16)) (and that the same can be said for the correlation coefficients of all its 2×2 submatrices), yet the B_1 of (18) has been shown here to *not* be positive definite. Thus, the test of Section III is deficient or fallacious since the matrix P_1 of (23) satisfies all the conditions of this test of the form of (12) (or (13)) yet is not positive definite. As with the principle minor test for positive semidefiniteness, this test is also necessary but not sufficient and a matrix satisfying this test doesn't make it even close to having the desired property. This test can only rigorously be perceived as an initial weeding out of some undesirable matrices only if they are really grossly off.

VI. COUNTEREXAMPLE TO METHODOLOGY OF SECTION IV

In general, the definition of (15) is equivalent to

$$p_{ik}^* \leftarrow \rho_{ik}^2 \tag{24}$$

by the definition of (4) substituted into (15). Further, the test of (16) should theoretically always be satisfied now that we can see via (7) (squared and rearranged) that it is a vacuous test. However, to proceed further and give the approach of Section IV the benefit of the doubt for the moment, suppose that for a particular computed covariance matrix P so much roundoff and truncation error has accrued to the extent that even the fairly loose theoretical condition of (7) fails to be satisfied (as embodied in the equivalent (16) failing to be satisfied); then according to the subsequent construction recipe of (17):

$$\begin{aligned} p_{ik}^* \leftarrow \frac{p_{ik}}{\sqrt{p_{ik}^*} + 0.001} &= \frac{\rho_{ik}\sigma_i\sigma_k}{\sqrt{\frac{p_{ik}^2}{p_{ii}p_{kk}}} + 0.001} \\ &= \frac{\rho_{ik}\sigma_i\sigma_k}{\sqrt{\rho_{ik}^2} + 0.001} = \frac{\rho_{ik}\sigma_i\sigma_k}{|\rho_{ik}| + 0.001} \\ &= \left[\frac{\rho_{ik}}{|\rho_{ik}| + 0.001}\right]\sigma_i\sigma_k \end{aligned} \tag{25}$$

and from the last expression in (25) it is easily recognized that the quantity within the braces is of a magnitude less than 1 and greater than -1 (independent of the actual entries of P) and therefore could be a valid correlation coefficient. Since this COVCHK methodology is evidently not sufficiently discriminating, every symmetric matrix satisfies these later stages of the COVCHK approach, even the P_1 of (23) defined in terms of the specific B_1 of (18).

Yet P_1 (and B_1) is definitely structurally undesirable to propagate as a covariance matrix since it is not even positive semidefinite. Indeed, a further disaster that can be explicitly demonstrated is that this B_1 of (18) (or P_1 of (23)) would *not* be revealed as being structurely undesirable by the COVCHK test of Section IV since it would go unaltered pass all the conditions of (14)–(16) and would not even activate the more extreme compensation mechanism of (17), and in conclusion completely satisfies the critical condition of (12) in Section III that is invoked last. Thus, this single counterexample demonstrates that the underlying theory behind the software logic used in COVCHK is deficient by not accomplishing its intended purpose as stated in Section IV. The mathematical significance of the matrix alterations described in (14)–(17) is apparently an engineering quick-fix (without finesse) in an attempt to maintain positive definiteness of the covariance matrices being provided computationally. A major objection is that the assignments of either (15) or (17) considerably alter the matrix beyond what would be considered to be reasonable limits that would normally be deemed acceptable as preserving any semblance of the original matrix. To further illustrate this charge, consider how the redefinition of (15) affects the B_1 of (18) to yield

$$B_1^* = \begin{bmatrix} 1 & \frac{1}{4} & \frac{9}{16} \\ \frac{1}{4} & 1 & \frac{1}{4} \\ \frac{9}{16} & \frac{1}{4} & 1 \end{bmatrix}. \quad (26)$$

This result *is* positive definite but is completely different from the original B_1 since the off-diagonal terms are smaller (as the square of the original terms) and the negative signs are lost.

VII. SUMMARY/CONCLUSIONS/OUTLOOKS

The status of computational tests for establishing matrix positive semidefiniteness and positive definiteness were reviewed in Section I. Two pervasive real-time "tests" were described in Sections III and IV that have been used for many years in varied applications in an attempt to ensure that computed covariances encountered in Kalman filter applications are positive definite. Structural representations of covariance matrices are reviewed in Section II as a prelude to constructing the counterexample and further demonstrating in Sections V and VI that it refutes, respectively, the two different approaches discussed, respectively, in Sections III and IV; so these unfortunately were bogus approaches despite the fact that they are pervasive.

Such bogus tests as these evidently arose as an attempt to fill the need for a quick check (over the entire mission time) of the massive number of matrices computationally encountered in real-time applications. All early Kalman filter implementations prior to the development of Bierman's so-designated $U-D-U^{\mathrm{T}}$ filter formulation ([16]) needed such monitoring since they were not numerically or computationally stable algorithms and, as such, tended to eventually become ill conditioned. As is now well known, the $U-D-U^{\mathrm{T}}$ filter *is* numerically stable and can be run for extremely long time periods or mission times without exhibiting any ill-conditioning of the covariance (or constituent components U and D) as previously requiring the quick fixes of Sections III and IV to "check" and "remedy". Unfortunately, the $U-D-U^{\mathrm{T}}$ filter formulation is not as universally used as it should be since there are "application bastions" that resist change[2] (even for the better) due to the vagaries of cost and tradition, so versions of the bogus definiteness/semidefiniteness tests can still be found in use. This Correspondence is to alert users to the potential dangers of using these nonrigorous tests.

For situations other than Kalman filtering where online tests of positive definiteness may still be desired for computed symmetric covariance matrices, determinant evaluations of Sylvester's criterion can be coded in hard-wired form for applications involving dimensions three (or less) using the "basket rule" ([17, ex. 17, pp. 66-67]), but use of the SVD for this purpose is definitely the preferred test for general dimension n, as already addressed in [1], [7]. Details for computationally handling SVD are addressed in [18, sect. 3].

As discussed in Section I, the unadorned SVD of Aasen's method is of order $n^3/6$, which may be too time consuming an operation for real-time use! The bogus test of Section III that is used in many real-time applications is of order n^2, so presumably if a version of SVD could also be made to be of order n^2, then it would ostensibly serve as a now rigorous algorithmic replacement with no greater penalty in computation time than already exists. (Moreover, the combinatorics-based operations count of Section III being $\frac{3}{2}n^2$ as compared with the $n^3/6$ of Aasen's SVD method indicates that use of Aasen's method is a lesser computational burden for applications with state sizes up to 18!) The only way to conveniently reduce a power of n in the fundamental SVD algorithm is to distribute the computations across n nearest-neighbor processors, as occurs in systolic array implementations. The good news is that such a recasting of SVD in terms of systolic arrays has already been initiated by others (as addressed in [19], [20]). Not to be overlooked, an alternative approach ([21]) is to use the systolic version of a QR algorithm to reveal the rank of a matrix. Other modern control applications that will also benefit from such a recasting of SVD beyond the positive definiteness testing addressed here are discussed in [22], [23].

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[2]While it is more challenging to understand the inner workings of the $U-D-U^{\mathrm{T}}$ version (as compared with that of a conventional Kalman filter), there is an easy answer to how to validate that a replacement $U-D-U^{\mathrm{T}}$ version is implemented correctly in software code. Just compare outputs under common identical test conditions with those of the standard Kalman filter that previously existed for the application. The outputs should be identical in the near term but the $U-D-U^{\mathrm{T}}$ should surpass the other for correctness in the long term.

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On Lyapunov Stability Analyses of the Switching Regulator/Filter System

The stability design criteria for the fourth-order switching regulator/low-pass filter system are obtained directly from the solution to the resultant simultaneous equations of the Lyapunov matrix equation.

Employing the matrix formulation of the second method of Lyapunov [1], a recent paper [2] performed a stability analysis on the fourth-order switching regulator/low-pass filter system, with the switching regulator modeled as a "linearized" negative resistance. This analysis included an examination of the Lyapunov matrix equation:

$$\mathbf{Q} = -(\mathbf{A}^T\mathbf{B} + \mathbf{BA}) \tag{1}$$

where **Q** is symmetric positive semidefinite, **B** is symmetric positive definite, and **A** is constant and

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