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Integral Evaluations Enabling Performance Tradeoffs for Two-Confidence-Region-Based Failure Detection

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I. Introduction

A DETAILED derivation of the equations is provided here, which subsequently allows rigorous setting of decision thresholds for the two-confidence-region (CR2) approach to failure detection. The CR2 approach itself is defined and described analytically in Ref. 1, with statistics derived in Ref. 2 and further refined in Refs. 3 and 4. This CR2 approach (for one and two dimensions) was historically used onboard U.S. submarines in monitoring the ships inertial navigation system (SINS)/electrostatically supported gyro monitor (ESGM) inertial navigation systems (INS) to detect the presence of ramp failures, a prominent failure mode observed to sometimes occur within the ESGM as it was originally being introduced onboard these boats 30+ years ago. The hybrid SINS/ESGM navigation system consisted of the existing SINS, an INS with conventional mechanically rotating gyros and having a second redundant SINS as a warm standby system,[5] in conjunction with the newer ESGM. The ESGM was generally more accurate (but initially more susceptible to failure).

In general, failure detection requires continuous vigilant monitoring of the observable output variables of the system. Under normal conditions, the output variables follow known patterns of evolution within certain limits of uncertainty introduced by slight random system disturbances and measurement noise in the sensors. When failures occur, the output variables deviate from their nominal state-space trajectories or evolutionary pattern. Failure-detection techniques are based on "spotting these deviations from the usual in the observable output variables" in some way.

Failure detection is a challenging event-detection problem that has been receiving attention only since the late 1960s, a representative sampling being Refs. 1–4 and 6–12 and their bibliographies, especially the extensive one in Ref. 10. For airborne navigation systems, Ref. 10 goes further by combining failure detection with the idea of using decentralized Kalman filters by including and incorporating existing, somewhat autonomous subsystems to offer a hierarchy of operational performance and accuracies that includes fail-safe behavior for a "limp-home" minimum capability. The problem of tracking a maneuvering target by radar or by optics (or by other similar acoustic transducers) is mathematically dual to that of failure detection.[13]

The CR2 approach to failure detection places one ellipsoidal confidence region about the nominally unfailed trajectory corresponding to the H_0 hypothesis and a second ellipsoidal confidence region about the Kalman estimate based on processing the actual measurements. When these two confidence regions are disjoint, implying a non-H_0 situation, then a failure is declared. The recent alternate more general test for ellipsoid overlap in Ref. 14, based on numerically solving the symmetric generalized eigenproblem using a Choleski factorization and the symmetric QR algorithm because $\lambda A \underline{x} = B\underline{x} \Leftrightarrow [\lambda A - B]\underline{x} = 0 \Rightarrow \underline{x}^T[\lambda A - B]\underline{x} = 0 \Leftrightarrow \underline{x}^T A[\lambda I - A^{-1}B]\underline{x} = 0$ (the last being the cornerstone of Ref. 14) as a simplifying relationship revealed in Ref. 15, can also benefit from these earlier CR2 results.

The associated underlying integral evaluation, examined in detail here starting from Eq. (29), enables a statistical analysis for the CR2 approach involving calculation of this detection algorithm's fundamental performance tradeoff characterized by what is known as receiver operating characteristics (ROC),[16] from which the operating point is set or fixed by specifying the value of the decision threshold to be used. The underlying mathematical evaluations conveyed herein are exercises in appropriately substituting variables and evaluating several intermediate integrals in creative ways and in constructively applying Cauchy's residue theorem, and, although this last step is initially somewhat tedious and unwieldy, we feel that it still warrants documenting because evidently others also seek to use it (as identified and commented on in Refs. 7–9). The CR2 failure-detection approach is predicated on the system of interest being adequately described or modeled in continuous time as a state variable representation [Ref. 17, Eq. (4–39)]:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x,t) + g(x,t)w(t) \tag{1}$$

along with discrete-time sensor data measurements being available of the following algebraic form:

$$z(t_k) = h[x(t_k), t_k] + v(t_k) \tag{2}$$

where $w(t)$ and $v(t)$ in the preceding are independent zero-mean Gaussian white noises of known, specified covariance intensity Q and R, respectively, and also independent of the Gaussian initial condition, where the associated noise covariance intensity matrices Q and R are symmetric and, respectively, positive definite and positive semidefinite and can be time varying. The initial covariance must also be symmetric and positive definite because the initial condition is $x(t_0) \sim \mathcal{N}[\bar{x}(0), P_0]$. Technical regularity conditions require that observability and controllability be satisfied by the system and its noises in Eqs. (1) and (2) or, at least, less restrictive conditions of detectability and stabilizability. The functions $f(x,t)$ and $g(x,t)$ are assumed to be bounded and measurable and satisfy a global Lipschitz condition, and $h[x(t),t]$ is continuous in x and t. A failure is modeled as an additional term, $+\nu\delta(t - t_f)$, on the right-hand side of Eq. (1), where t_f is the initially unknown time of failure onset and the initially unknown vector ν conveys the consequential effect of the failure on the other states of the system and $\delta(\cdot)$ is the Dirac delta function.

The system is assumed to be outfitted or equipped with an adequate, perhaps reduced-order, Kalman filter[18] matched to a linearized version of the system. For INS involving a constellation

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of gyros and accelerometers, even though the mechanization itself is nonlinear (e.g., space stable, local level, or strapdown), the underlying error model is linear,[19] and, as such, possesses an optimal estimator that can be obtained as the output of a linear Kalman filter, usually implemented in indirect feedback form (Ref. 17, Chap. 6).

The CR2 algorithm itself makes use of a proper subset of the state estimates $\hat{x}$ and corresponding associated subset of the covariance of estimation error $P_1(t)$ (computed from the underlying discrete-time matrix Riccati equation), which are available as outputs online from the Kalman filter at each discrete time step k. The subset constitutes the particular states being monitored for failures. For the ESGM, the states of interest were the three INS gyro drift rates being monitored for the presence or absence of a gyro ramp failure. The covariance output of an associated matrix Lyapunov equation $P_2(t)$ (corresponding to the covariance or uncertainty of the system without any sensor measurement being available) is also assumed to be available within the CR2 algorithm.

II. Overview of the CR2 Approach to Failure Detection

At a particular fixed discrete time step k, the defining equations that need to be evaluated for the completely general n-dimensional case in order to specify the probability of false alarm $P_{\text{fa}}(k)$ and probability of correct detection $P_d(k)$ are provided in Ref. 2. Both are expressed in terms of the underlying Gaussian densities assumed to be present under the simple hypotheses H_0 and mixed hypothesis H_1 and are also expressed in terms of the measured signal-to-noise ratio, having the following structural form:

$$\text{SNR}(k) \triangleq \sqrt{\boldsymbol{d}^T(k) P_{\tilde{x}\tilde{x}}^{-1}(k)\boldsymbol{d}(k)} \tag{3}$$

and in terms of the computed scalar CR2 test statistic and in its relationship to the [possibly time-varying decision threshold $K_1(k)$], where in the preceding:

$$\boldsymbol{d}(k) \triangleq \text{the mean deterministic response of the Kalman filter to an hypothesized specific failure mode } \bar{\boldsymbol{\nu}} \tag{4}$$

where the time evolution of the vector d(k) can be evaluated explicitly from a joint system and filter simulation using the truth models for system and Kalman filter but with the system and measurement noise sample functions zeroed out (i.e., $Q \equiv 0$, $R \equiv 0$) and only $\bar{\boldsymbol{\nu}}$ as the particular failure mode of interest being activated. Specification of the particular system error modes of concern should be available from a failure modes and effects analysis.

The following quantities are fundamental in detection theory and need to be evaluated for all practical applications. We treat the evaluation of $P_{\text{fa}}(k)$ and $P_d(k)$ separately in what follows, corresponding to the different form of the underlying probability distribution functions (PDF) under H_0 and H_1, respectively, as

$$\text{(no-failure)} \qquad \text{H}_0 : \tilde{x}_i(k) \sim \mathcal{N}(0, [P_{\tilde{x}\tilde{x}}(k)]_{ii}) \qquad \text{(zero mean)} \tag{5}$$

$$\text{(failure)} \qquad \text{H}_1 : \tilde{x}_i(k) \sim \mathcal{N}(d(k), [P_{\tilde{x}\tilde{x}}(k)]_{ii}) \qquad \text{(nonzero mean)} \tag{6}$$

where, in the preceding:

$$\tilde{\boldsymbol{x}}(k) = \boldsymbol{x}(k) - \hat{\boldsymbol{x}}(k) \tag{7}$$

$$\hat{\boldsymbol{x}}(k) \triangleq E[\boldsymbol{x}(k)|Z(k)] \text{ (the conditional mean being the optimal estimate}^{17}\text{)} \tag{8}$$

$$E[\boldsymbol{x}(k)] = E\left[E[\boldsymbol{x}(k)|Z(k)]\right], \text{ (a property of conditional expectation)} \tag{9}$$

$$\begin{aligned} E[\tilde{\boldsymbol{x}}(k)] &= E[\boldsymbol{x}(k)] - E[\hat{\boldsymbol{x}}(k)] = E[\boldsymbol{x}(k)] - E\left[E[\boldsymbol{x}(k)|Z(k)]\right] \\ &= E[\boldsymbol{x}(k)] - E[\boldsymbol{x}(k)] = 0 \end{aligned} \tag{10}$$

$$\begin{aligned} 0 &= E[\tilde{\boldsymbol{x}}(k)\hat{\boldsymbol{x}}^T(k)] = E[\boldsymbol{x}(k)\hat{\boldsymbol{x}}^T(k)] - E[\hat{\boldsymbol{x}}(k)\hat{\boldsymbol{x}}^T(k)] \\ &= E[\boldsymbol{x}(k)\hat{\boldsymbol{x}}^T(k)] - P_1(k) \end{aligned} \tag{11}$$

which, when Eq. (7) is multiplied by its own transpose and unconditional expectations taken throughout, yields:

$$P_{\tilde{x}\tilde{x}}(k) = P_2(k) - P_1(k) \tag{12}$$

where the cross terms dropped out as a consequence of the Hilbert space projection theorem result of the left-hand side of Eq. (11). The following has been rigorously established earlier in Ref. 1, lemma 5.1 by taking the synchronous difference between the two respective matrix difference equations, Lyapunov and Riccati, which describe their evolution (in discrete time) by demonstrating that the difference is always positive definite as it evolves in time:

$$P_2(k) - P_1(k) > 0 \qquad \text{for all} \qquad k > 0 \tag{13}$$

In the preceding expressions of conditional expectation, the notation $Z(k)$ denotes the sigma algebra generated by the sequence of measurements received $z(i)$ for $0 \le i \le k$ (also interpreted as the subspace spanned by the discrete measurements received up to time k).

A mechanization of a failure-detection solution using the CR2 approach is of the following form:

Decide that no failure occurred (indicative of H_0 holding) at time step k, when:

$$\ell(k) \le K_1(k) \tag{14a}$$

Decide that a failure occurred (indicative of H_1 holding) at time step k, when:

$$\ell(k) > K_1(k) \tag{14b}$$

III. Evaluation of $P_{\text{fa}}(k)$ for the CR2 Detection Test

For CR2, a detection is declared when the scalar test statistic $\ell(k)$ exceeds the decision threshold setting $K_1(k)$ as depicted in Eq. (14) so that the probability of false alarm corresponds to the following situation for the n-dimensional case:

$$\begin{aligned} P_{\text{fa}}(k) &\triangleq Prob[\ell(k) > K_1(k)|\text{H}_0] \\ &= \int \cdots \int_{\ell(k) > K_1(k)} \mathcal{N}[\mathbf{0}, P_{\tilde{x}\tilde{x}}(k)]\, \mathrm{d}\tilde{\boldsymbol{x}} \end{aligned} \tag{15}$$

The preceding expression is difficult to evaluate for general dimension n. However, it is relatively easy to evaluate for the scalar case (in Sec. III.A) but very challenging even for the case of $n = 2$ (in Sec. III.B). In both cases, simpler expressions are needed for the test statistic in order to enable explicit evaluation of the integrals encountered corresponding to Eq. (15) and to ultimately enable specification of the requisite decision threshold $K_1(k)$, which corresponds to a value of $P_{\text{fa}}(k)$ imposed as a constraint to satisfy system performance specifications.

For the SSBN ESGM application of interest that funded this investigation,[20] only gyros and accelerometers with one or two input axes are involved so that the corresponding version of CR2 only needs one- and two-dimensional CR2 mechanizations, respectively, to monitor their behavior. We therefore restrict attention here to evaluating the $P_{\text{fa}}(k)$ and $P_d(k)$ for just the one- and two-dimensional cases as the simplification in vogue rather than pursue the more general n-dimensional case (which remains an open question for later generations to tackle and solve). In a three-dimensional world, the ESGM had two gyros, each with two input axes, one of the four input axes being redundant, so that the gyro with nonredundant input axes (i.e., both input axes participating in the computed navigation solution) used a two-dimensional version of CR2 and the other gyro (with only a single actively used input axis participating in the computed navigation solution) needed to be outfitted with only a one-dimensional version of CR2. The statistical analysis and calculation of the scalar CR2 test statistic for the two-dimensional case is much harder to handle than that for the one-dimensional case, as will become quite evident in Secs. III.B, III.C, and IV.B.

A. CR2 $P_{\text{fa}}(k)$ Calculations Simplify Nicely for the Scaler Case

The constrained optimization problem and associated scalar Lagrange multiplier that define the scalar CR2 test statistic both have a closed-form exact solution for the one-dimensional case as, respectively, Eqs. (20) and (21) of Ref. 2. From Ref. 2, Eq. (23), the scalar CR2 test statistic for the one-dimensional case is

$$\ell(k) \triangleq \frac{[\hat{x}_i(k)]^2}{\left[\sqrt{[P_2(k)]_{ii}} + \sqrt{[P_1(k)]_{ii}}\right]^2} \tag{16}$$

which, when substituted into Eq. (15), yields

$$\begin{aligned} P_{\text{fa}}(k) &= \int_{\hat{x}_i^2 > K_1(k)\cdot(\sqrt{P_2}+\sqrt{P_1})^2} \mathcal{N}_{\hat{x}}(0, [P_2(k) - P_1(k)])\,\mathrm{d}\hat{x}_i \\ &= \int_{u^2 > K_1(k)\cdot(\sqrt{P_2}+\sqrt{P_1})/(\sqrt{P_2}-\sqrt{P_1})^2} \exp\left[\frac{-u^2}{\sqrt{\pi}}\right]\mathrm{d}u \\ &= 1 - \frac{1}{2}\mathrm{erf}\left\{\sqrt{\frac{K_1(k)}{2}}\cdot\sqrt{\frac{\left[\sqrt{P_2(k)}+\sqrt{P_1(k)}\right]}{\left[\sqrt{P_2(k)}-\sqrt{P_1(k)}\right]}}\right\} \end{aligned} \tag{17}$$

To obtain the decision threshold $K_1(k)$, given a fixed value of $P_{\text{fa}}(k)$ to be maintained at each check time, involves using tables to solve for the constant b in the following equation:

$$P_{\text{fa}}(k) = 1 - \tfrac{1}{2}\mathrm{erf}\left[b\big/\sqrt{2}\right] \tag{18}$$

which, by substituting as depicted in Ref. 2, Eq. (39a), the CR2 decision threshold for the one-dimensional case is

$$K_1(k) = b^2 \cdot \left\{\frac{\left[\sqrt{[P_2(k)]_{ii}} - \sqrt{[P_1(k)]_{ii}}\right]}{\left[\sqrt{[P_2(k)]_{ii}} + \sqrt{[P_1(k)]_{ii}}\right]}\right\} \tag{19}$$

The preceding is a time-varying decision threshold that can be used to maintain a constant specified instantaneous false alarm rate. (A methodology is provided in Ref. 3 for specifying a decision threshold using random process level-crossing theory so that a particular probability of false alarm exists over an entire specified time interval and not just instantaneously at each discrete check time k.) Real-time online mechanization of CR2 for one dimensions uses only Eqs. (16), (18), and (19) and the two comparison tests of Eq. (14).

B. Evaluating CR2 $P_{\text{fa}}(k)$ for the Challenging Two-Dimensional Case

The expression for the scalar CR2 test statistic for $n = 2$ is considerably more complex than for the one-dimensional case. It is obtained by first solving the scalar iteration equation for the associated Lagrange multiplier [Ref. 1, Eq. (34), Ref. 8, Eq. (1)]:

$$\lambda_{n+1} = 1\bigg/\left\{1 + \frac{\boldsymbol{w}^T(k)[(1-\lambda_n)P_2(k) + \lambda_n P_1(k)]^{-1}P_1(k)[(1-\lambda_n)P_2(k)+\lambda_n P_1(k)]^{-1}\boldsymbol{w}(k)}{\boldsymbol{w}^T(k)[(1-\lambda_n)P_2(k)+\lambda_n P_1(k)]^{-1}\boldsymbol{w}(k)}\right\} \quad \text{for} \quad \lambda_0 = 0.75 \tag{20}$$

where $\boldsymbol{w}(k) \triangleq \hat{\boldsymbol{x}}_1(k) - \bar{\boldsymbol{x}}(k)$. This iteration equation converges geometrically fast as a contraction mapping (Ref. 1, theorem 5.1) to a unique solution $\bar{\lambda}(k)$, which is then substituted back into the accompanying Lagrangian saddle point solution for the minimum $x^*(k)$ of the constrained optimization problem that serves as the scalar CR2 test statistic:

$$\ell(k) = \ell(\bar{\lambda}, \boldsymbol{x}^*(k)) = \bar{\lambda}(1-\bar{\lambda})[(1-\bar{\lambda})P_2(k) + \bar{\lambda}P_1(k)] \tag{21}$$

The preceding expression along with $K_1(k)$ is used in the limits of the integral representing the $P_{\text{fa}}(k)$. For the case of $n = 2$, the integrals of Eq. (15) become

$$P_{\text{fa}}(k) = \int_{K_1(k)}^{\infty} p_{L|\mathrm{H}_0}(\ell)\,\mathrm{d}\ell$$

$$= \int_{K_1(k)}^{\infty}\left[\frac{1}{|a_1|}p_{\chi^2}\left(\frac{\cdot}{a_1}\right) * \frac{1}{|a_2|}p_{\chi^2}\left(\frac{\cdot}{a_2}\right)\right]\mathrm{d}L \tag{22}$$

$$= \int_{K_1(k)}^{\infty}\left[\frac{\exp(-L/2a_2)}{2\pi\sqrt{a_1a_2}}\int_0^L \frac{e^{-bx}}{\sqrt{x(L-x)}}\,\mathrm{d}x\right]\mathrm{d}L \tag{23}$$

$$= \int_{K_1(k)}^{\infty}\left[\left(\frac{1}{4\pi\sqrt{a_1a_2}}\right) \times \int_{-\pi}^{\pi}\exp\left\{-\frac{1}{2}\left[\frac{1}{a_2} + b + b\sin\theta\right]L\right\}\mathrm{d}\theta\right]\mathrm{d}L \tag{24}$$

$$= \left(\frac{1}{4\pi\sqrt{a_1a_2}}\right)\int_{-\pi}^{\pi}\left[\int_{K_1(k)}^{\infty} \times \exp\left\{-\frac{1}{2}\left[\frac{1}{a_2} + b + b\sin\theta\right]L\right\}\mathrm{d}L\right]\mathrm{d}\theta \tag{25}$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1a_2}}\right)\int_{-\pi}^{\pi}\left[\frac{\exp\{-(bK_1/2)\sin\theta\}}{1+(b/C)\sin\theta}\right]\mathrm{d}\theta \tag{26}$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1a_2}}\right)\int_{-\pi}^{\pi}\left[\sum_{i=0}^{\infty}\frac{(-bK_1/2)^i(\sin\theta)^i/i!}{1+(b/C)\sin\theta}\right]\mathrm{d}\theta \tag{27}$$

$$= \left(\frac{\exp\{-K_1C/2\}}{2\pi C\sqrt{a_1a_2}}\right)\sum_{i=0}^{\infty}\left[(-bK_1/2)^i/i!\right] \cdot \int_{-\pi}^{\pi}\frac{(\sin\theta)^i}{1+(b/C)\sin\theta}\,\mathrm{d}\theta \tag{28}$$

$$= \left(\frac{\exp\{-K_1C/2\}}{\pi C\sqrt{a_1a_2}}\right)\sum_{i=0}^{\infty}[(K_1b)^i/i!] \times \int_{-\infty}^{\infty}\frac{x^i}{\{[x^2+2(b/C)x+1](1+x^2)^i\}}\,\mathrm{d}x \tag{29}$$

$$= \left(\frac{\exp\{-K_1C/2\}}{e_0}\right)\left[e_0 + e_1K_1 + e_2K_1^2 + e_3K_1^3 + e_4K_1^4 + e_5K_1^5 + \cdots\right] \tag{30}$$

where in the preceding [Ref. 2, Eqs. (B.1-11–B.1-13)]:

$$a_i > 0 \qquad \text{for} \qquad i = 1, 2 \tag{31}$$

$$b \triangleq \frac{a_2 - a_1}{2a_2a_1} > 0 \tag{32}$$

$$C \triangleq \frac{1}{a_2} + b = \frac{1}{a_2} + \frac{a_2 - a_1}{2a_2a_1} = \frac{a_2 + a_1}{2a_2a_1} > 0 \tag{33}$$

$$C > b \tag{34}$$

$$1 > (b/C)^2 > 0 \tag{35}$$

and $*$ denotes the operation of convolution. Because the integrand of Eq. (22) is positive as are those of Eqs. (23) and (24) as PDFs and, as such, have a finite integral when integrated over $K_1(k)$ to ∞, use of Fubini's theorem[21] allows the rigorous interchange of the order of integration, resulting in Eq. (25). [The inner integral in Eq. (24) was originally to be integrated from $-\pi/2$ to $\pi/2$, but that is equivalent to the more convenient version from $-\pi$ to π when divided by two.] After integrating out the variable L in Eq. (25) to obtain Eq. (26), Eq. (26) can be rewritten using the series expansion of the exponential, as in Eq. (27). Because the resulting series of continuous functions in Eq. (27) is a uniformly convergent series by the Weierstrass M-test,[22] the order of integration and summation can be rigorously interchanged in Eq. (27) to result in Eq. (28). Using the half-angle substitution[23] of $x = \tan(\theta/2)$ for which $\theta = 2\arctan x$ and $\mathrm{d}\theta = 2\,\mathrm{d}x/1 + x^2$ in Eq. (28) yields Eq. (29). Going from Eq. (29) to obtain the result of Eq. (30) is very challenging and tedious because it involves fairly long intermediate expressions, but they eventually collapse into shorter more manageable expressions, as summarized next.

C. Obtaining a Tractable Series Needed for Handling the Two-Dimensional Case of CR2

The following integral that arose as Eq. (29)

$$P_{\mathrm{fa}}(k) = \left(\frac{\exp\{-K_1C/2\}}{\pi C\sqrt{a_1a_2}}\right)\sum_{i=0}^{\infty}\left[(K_1b)^i/i!\right] \times \int_{-\infty}^{\infty}\frac{x^i}{[(x^2+2(b/C)x+1)(1+x^2)^i]}\,\mathrm{d}x \tag{36}$$

can be evaluated over a closed path involving a semicircle and the real axis in the complex plane using the Cauchy residue theorem[24] in conjunction with some limiting arguments to make the upper-half-plane semicircle have a radius that goes to infinity (and the corresponding real axis segment go from $-\infty$ to ∞), as explained next.

Notice that the general integral of Eq. (29) has poles at the values of z, where the following two polynomials have zeros:

$$0 = z^2 + 2(b/C)z + 1 \tag{37}$$

$$0 = (z^2+1)^i \tag{38}$$

specifically, the roots of interest, which fall within a closed infinite semicircle in the upper-half complex plane, occur at the following values of z:

$$z = -b/C + \jmath\sqrt{1-(b/C)^2} \tag{39}$$

$$z = +\jmath\,(\text{of multiplicity } i) \tag{40}$$

and the negative imaginary roots of both of these quadratic polynomials lie outside of the closed upper semicircle and therfore play no role in the numerical evaluation via a sum of the enclosed residues in the counterclockwise direction. The contribution of the path integral along the infinite semicircle is zero because the degree of the denominator is more than two greater than that of the numerator so that the real integrals of Eqs. (29) and (36) are equivalent to the complex path integrals over the simply connected region enclosed:

$$\begin{aligned}2\pi\jmath\sum\mathrm{Res} &= \oint\frac{z^i\,\mathrm{d}z}{[(z^2+2(b/C)z+1)(1+z^2)^i]}\\ &= \lim_{R\to\infty}\int_0^{\pi}\frac{\jmath R^{i+1}e^{\jmath(i+1)\theta}\,\mathrm{d}\theta}{[[R^2e^{\jmath2\theta}+2(b/C)Re^{\jmath\theta}+1](1+R^2e^{\jmath2\theta})^i]}\\ &\quad+\lim_{R\to\infty}\int_{-R}^{+R}\frac{x^i\,\mathrm{d}x}{[[x^2+2(b/C)x+1](1+x^2)^i]}\\ &= 0+\int_{-\infty}^{\infty}\frac{x^i\,\mathrm{d}x}{[[x^2+2(b/C)x+1](1+x^2)^i]}\end{aligned} \tag{41}$$

where the integrand is analytic within the contour described except at the just-mentioned simple poles.

Evaluation of the integrals of Eq. (36) for the first six terms, using the Cauchy residue theorem, yields the first three easy evaluations that demonstrate how the evaluations will be performed next for the remaining three harder cases:

$$\begin{aligned}I_0 &\triangleq \int_{-\infty}^{\infty}\frac{\mathrm{d}x}{\{[x^2+2(b/C)x+1]\}} = 2\pi\jmath\cdot\frac{1}{\jmath2\sqrt{1-(b/C)^2}}\\ &= \frac{\pi}{\sqrt{1-(b/C)^2}}\end{aligned} \tag{42}$$

$$\begin{aligned}I_1 &\triangleq \int_{-\infty}^{\infty}\frac{x\,\mathrm{d}x}{\{[x^2+2(b/C)x+1](1+x^2)\}}\\ &= 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}(-2b/C)}+\frac{1}{2(2b/C)}\right]\end{aligned} \tag{43}$$

$$\begin{aligned}I_2 &\triangleq \int_{-\infty}^{\infty}\frac{x^2\,\mathrm{d}x}{\{[x^2+2(b/C)x+1](1+x^2)^2\}}\\ &= 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-b/C)^2}\right]\\ &\quad+2\pi\jmath\left(\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{x^2}{[x^2+2(b/C)x+1](x+1\jmath)^2}\right\}\right)\Bigg|_{x=1\jmath}\\ &= 2\pi\jmath\left[\frac{1}{2\jmath\sqrt{1-(b/C)^2}(-b/C)^2}\right]\\ &\quad+2\pi\jmath\left\{\frac{-2x^4-2(b/C)x^3+\jmath[2(b/C)x^2+2x]}{[x^2+2(b/C)x+1](x+1\jmath)^3}\right\}\Bigg|_{x=1\jmath}\\ &= 2\pi\left[\frac{1}{2\sqrt{1-(b/C)^2}(2b/C)^2}-\frac{1}{2(2b/C)^2}\right]\end{aligned} \tag{44}$$

Continuing in like manner, but sparing the reader much of the long unwieldy intermediate expressions, yields

$$\begin{aligned}I_3 &\triangleq \int_{-\infty}^{\infty}\frac{x^3\,\mathrm{d}x}{\{[x^2+2(b/C)x+1](1+x^2)^3\}}\\ &= 2\pi\left\{\frac{1}{2\sqrt{1-(b/C)^2}(-2b/C)^3}+\frac{1}{2}\left[\frac{1}{(2b/C)^3}+\frac{1}{8(2b/C)}\right]\right\}\end{aligned} \tag{45}$$

$$\begin{aligned}I_4 &\triangleq \int_{-\infty}^{\infty}\frac{x^4\,\mathrm{d}x}{\{[x^2+2(b/C)x+1](1+x^2)^4\}}\\ &= 2\pi\left\{\frac{1}{2\sqrt{1-(b/C)^2}(-2b/C)^4}-\frac{1}{2^4}\left[\frac{1}{(2b/C)^2}+\frac{8}{(2b/C)^4}\right]\right\}\end{aligned} \tag{46}$$

$$\begin{aligned}I_5 &\triangleq \int_{-\infty}^{\infty}\frac{x^5\,\mathrm{d}x}{\{[x^2+2(b/C)x+1](1+x^2)^5\}}\\ &= 2\pi\left\{\frac{1}{2\sqrt{1-(b/C)^2}(-2b/C)^5}+\frac{1}{2^4}\left[\frac{1}{(2b/C)^5 2}\right.\right.\\ &\quad\left.\left.+\frac{1}{(2b/C)^3 2^4}+\frac{3}{(2b/C)2^8}\right]\right\}\end{aligned} \tag{47}$$

Although the preceding results were originally derived by long hand, they fortuitously possessed a type of internal error cross check by the imaginary contribution of the enclosed residues collapsing to be identically zero.

As derived and defined in Ref. 2 [lemma 1, Eq. (B.1-2)], a useful auxiliary matrix is

$$S(\bar{\lambda}, k) \triangleq [P_2(k) - P_1][(1-\bar{\lambda})P_2(k) + \bar{\lambda}P_1] \tag{48}$$

which can be used as an intermediary in specifying the requisite parameters a_1, a_2, b^2, and C, from which ultimately the parameters $e_0, e_1, e_2, e_3, e_4, e_5$ are defined [Ref. 2, Eqs. (41–44)]. It is also established in Ref. 2, [Eqs. (B.2-12–B.2-13)], by the simple algebraic manipulation of inequalities, that

$$C > 0 \tag{49}$$

$$e_i > 0 \qquad \text{for all} \qquad i = 0, \ldots, 5 \tag{50}$$

To solve Eq. (30) for the unknown $K_1(k)$, a useful contrivance is to decompose it into two separate algebraic equations to be solved simultaneously as

$$y_1(K_1) \triangleq e_0 + e_1K_1 + e_2K_1^2 + e_3K_1^3 + e_4K_1^4 + e_5K_1^5 \tag{51}$$

$$y_2(K_1) \triangleq e_0 P_{\mathrm{fa}}(k) \cdot \exp[CK_1/2] \tag{52}$$

Notice that the vertical intercept of the two equations is ordered as $e_0 > P_{\mathrm{fa}} \cdot e_0$, and the exponentially increasing term initially starts below the quintic at $K_1 = 0$ but ultimately grows to intersect it because the exponent is purely positive and the exponential will eventually dominate the quintic polynomial even though it starts below it. A successive approximations implementation of these two equations[25] can be used to easily solve this problem evaluation for the unknown $K_1(k)$, as depicted in Ref. 2 (Fig. 4). Convergence is obvious from the figure cited. This successive approximations approach is iterated to convergence for each successive k to yield a time-varying decision threshold that yields a constant false alarm rate (CFAR) implementation of the CR2 test. Real-time online mechanization of CR2 for two-dimensions uses only Eqs. (20), (21), (51), and (52) and the two comparison tests of Eq. (14). The best grouping to minimize the associated computational burden in terms of operation counts is also identified in Refs. 1 and 8 [Eqs. (4) and (5)].

IV. Evaluation of $P_d(k)$ for the CR2 Detection Test

For CR2, the probability of correct detection corresponds to the following situation for the n-dimensional case:

$$P_d(k) \triangleq Prob[\ell(k) > K_1(k)|\mathrm{H}_1]$$

$$= \int \cdots \int_{\ell(k) > K_1(k)} \mathcal{N}(\boldsymbol{d}(k), P_{\tilde{x}\tilde{x}}(k))\, \mathrm{d}\tilde{\boldsymbol{x}} \tag{53}$$

Similar to the situation for evaluation of $P_{\mathrm{fa}}(k)$, the preceding expression is difficult to evaluate for general dimension n. However, it is easy to evaluate for the scalar case (in Sec. IV.A) and tractable but more challenging for the case of $n = 2$ (in Sec. IV.B).

A. CR2 $P_d(k)$ Evaluation Simplifies Nicely for the Scaler Case

In complete analogy to what was done in Sec. III.A and the simplifications that accrued for the one-dimensional case, the integral of Eq. (53) reduces to the following closed form (with constituent parts that are known and easy to evaluate):

$$P_d(k) = 1 - \frac{1}{2}\mathrm{erf}\left\{\left[\frac{\mathrm{SNR}(k)}{\sqrt{2}}\right] + \sqrt{\frac{K_1(k)}{2}} \cdot \sqrt{\frac{\left[\sqrt{P_2(k)} + \sqrt{P_1(k)}\right]}{\left[\sqrt{P_2(k)} - \sqrt{P_1(k)}\right]}}\right\}$$

$$- \frac{1}{2}\mathrm{erf}\left\{\left[\frac{\mathrm{SNR}(k)}{\sqrt{2}}\right] - \sqrt{\frac{K_1(k)}{2}} \cdot \sqrt{\frac{\left[\sqrt{P_2(k)} + \sqrt{P_1(k)}\right]}{\left[\sqrt{P_2(k)} - \sqrt{P_1(k)}\right]}}\right\} \tag{54}$$

where in the preceding, the expression for the signal-to-noise ratio of Eq. (3) simplifies as

$$\mathrm{SNR}(k) = \frac{|d(k)|}{\sqrt{P_2(k) - P_1(k)}} \tag{55}$$

B. Evaluating CR2 $P_d(k)$ for the Challenging Two-Dimensional Case

For the two-dimensional case, after performing an offset by the indicated mean and scaling by the covariance matrix present in the Gaussian distribution, Eq. (53) simplifies as

$$P_d(k) = 1 - \int\int_G \mathcal{N}_u(\mathbf{0}, I)\, \mathrm{d}\boldsymbol{u} \tag{56}$$

where G is the following elliptical region:

$$\left[\boldsymbol{u} + [P_2 - P_1]^{-\frac{1}{2}}\boldsymbol{d}(k)\right]^T \bar{A}^{-1}(\bar{\lambda})$$

$$\times \left[\boldsymbol{u} + [P_2 - P_1]^{-\frac{1}{2}}\boldsymbol{d}(k)\right] \leq \frac{K_1(k)}{\bar{\lambda}(1-\bar{\lambda})} \tag{57}$$

where the integral here represents the volume under the circular (independent) bivariate Gaussian surface enclosed by an offset ellipse and can be evaluated using existing tables.[26] A circular approximation to the preceding elliptical region is offered in Ref. 2 [Eq. (54)] and enables these integrals of a circular bivariate Gaussian surface to be evaluated over an offset circle, as found in more generally available tables.[27,28] This completes how to characterize the CR2 performance in terms of its associated ROC, where the particular $P_{\mathrm{fa}}(k)$ and $P_d(k)$ associated with the operating point utilized [as defined by the particular threshold value $K_1(k)$] occur as parameters in the more realistic nonideal three-state switches used in associated system reliability/availability diagrams (e.g., Ref. 4) vice use of an ideal single-state switch [corresponding to $P_{\mathrm{fa}}(k) = 0$ and $P_d(k) = 1$]. The results of applying CR2 to real SINS/ESGM sensor data are depicted in Ref. 2 (Fig. 3) and in the predecessor contractor reports. Only failure magnitudes corresponding to $\mathrm{SNR} \simeq 12.5$ dBm or more above the background noise of the coarser SINS will have good detection behavior, a standard benchmark number in radar detection as well. A favorable aspect of this particular SINS/ESGM application is that deducing the exact time of ESGM gyro ramp failure onset is not particularly important because a ramp starts out small at its inception no matter what the magnitude of its slope and at that time does not have any adverse impact on total navigation accuracy; however, as time elapses, it will eventually become large enough to be detected as it exceeds the larger background noise of the SINS and is then a threat to the overall system accuracy that is to be protected by use of CR2 failure detection.

There are detailed figures intuitively depicting all aspects of CR2 diagrammatically in Refs. 1–3. The fundamental characterizations of CR2 found in Refs. 1 and 2 are particularly amenable to being visualized graphically because the underlying test for the overlap of two ellipsoidal confidence region sheaths at a particular check time k is geometrical in character and is solved by embedding the n-dimensional problem within an $(n+1)$-dimensional space. A recent test for n-dimensional ellipsoid overlap,[14] which avoids CR2's restrictive hypothesis as the condition of Eq. (13) and so is more generally appicable, also embeds the problem in $(n+1)$-dimensions in order to elegantly solve the general n-dimensional overlap problem, as pointed out in Ref. 15. This recent overlap test can also be applied within the more complete failure detection, identification, and reconfiguration methodology offered in Ref. 10, which utilizes decentralized Kalman filters, but which previously avoided using CR2 because it had its own overlap test of sorts that was coarser or less refined than what is now available by using the more general overlap test of Ref. 14.

V. Conclusions

We have summarized the rigorous mathematics underlying the CR2 approach to failure detection, with particular attention being given here to the evaluation of the complex integrals, which, up to now, had received short shrift in other associated CR2 documentation. This evaluation was crucial in order to evaluate $P_{fa}(k)$ from which $K_1(k)$ can then be specified and afterwards consequently firms up the associated $P_d(k)$, which all arises in characterizing the CR2 performance in terms of ROC (and CFAR values of P_{fa} and associated P_d incurred for failure magnitudes to be protected against, as are usually identified in an associated failure modes and effects analysis for the system of interest). We also reminded the reader that mathematical model structures identical to those encountered for failure detection arise in tracking uncooperative maneuvering or evading targets (using radar or passive or active optics or even sonar or other acoustic trackers).

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