…uth and

(A1)

…OS. The

(A2a)

(A2b)

…ight with …by V = …ω = col …ctor rate

(A3)

…the LOS …elevation …, respec-

(A4a)

(A4b)

…ng in the …D^*, S_{azim} …dent from …s that the …total LOS

(A5)

…path, i.e., …(A4a) that …mum, and …he loci of …found by …derivative …satisfy the

(A6)

…of minima

…$= S_{azim}/S_{elev}$ …on," and is

(A7)

…∞ the local …$-D^*/h$ =

Acknowledgments

This research has been supported by a research grant from NASA Ames Research Center, Aerospace Human Factors Division, Moffett Field, CA 94035, under cooperative agreement No. NAGW-1128. S. Hart and D. Foyle of Ames Research Center have been the Scientific Monitors for this grant.

References

[1]Gibson, J. J., *The Perception of the Visual World*, Houghton Mifflin, Boston, MA, 1950.

[2]Gibson, J. J., Olum, P., and Rosenblatt, F., "Parallax and Perspective During Aircraft Landings," *American Journal of Psychology*, Vol. 68, 1955, pp. 372–385.

[3]Gibson, J. J., "Visually Controlled Locomotion and Visual Orientation in Animals," *British Journal of Psychology*, Vol. 49, 1958, pp. 182–194.

[4]Gordon, D. A., "Static and Dynamic Visual Fields in Human Space Perception," *Journal of the Optical Society of America*, Vol. 55, 1965, pp. 1296–1303.

[5]Gordon, D. A., "Perceptual Basis of Vehicular Guidance," *Public Roads*, Vol. 34, 1966, pp. 53–68.

[6]Warren, R., "The Perception of Egomotion," *Journal of Experimental Psychology: Human Perception and Performance*, Vol. 2, No. 3, 1976, pp. 448–456.

[7]Warren, R., "Optical Transformation During Movement," Air Force Office of Scientific Research, Rept. AFOSR-TR-82-1028, NTIS AD-A122 275/1, Ohio State Univ., Columbus, OH, Oct. 1982.

[8]Zacharias, G. L., "Flow-Field Cueing Conditions for Inferring Observer Self-Motion," Bolt Beranek and Newman, Inc., Rept. 5118, Cambridge, MA, Sept. 1982.

[9]Zacharias, G. L., Caglayan, A. K., and Sinacori, J. B., "A Visual Cueing Model for Terrain-Following Applications," *Journal of Guidance, Control, and Dynamics*, Vol. 8, No. 2, 1985, pp. 201–207.

[10]Denton, G. G., "The Influence of Visual Pattern on Perceived Speed," *Perception*, Vol. 9, 1980, pp. 393–402.

[11]Owen, D. H., Wolpert, L., and Warren, R., "Effects of Optical Flow Acceleration, Edge Acceleration, and Viewing Time on the Perception of Egospeed Acceleration," NASA Scientific and Technical Information Facility, *1984, pp. 79–140.*

[12]Johnson, W. W., Bennett, C. T., O' Donnell, K., and Phatak, A. V., "Optical Variables Useful in the Active Control of Altitude," *Proceedings of the 23rd Annual Conference on Manual Control* (Cambridge, MA), June 1988.

[13]Awe, C. A., Johnson, W. W., and Schmitz, F., "Inflexibility in Selecting the Optical Basis for Perceiving Speed," *Proceedings of the Human Factors Society 33rd Annual Meeting*, 1989, pp. 1440–1444.

[14]Larish, J. F., and Flach, J. M., "Sources of Optical Information Useful for the Perception of Velocity of Rectilinear Selfmotion," *Journal of Experimental Psychology: Human Perception and Performance*, Vol. 16, No. 2, 1990, pp. 295–302.

[15]Naish, J. M., "Control Information in Visual Flight," Seventh Annual Conf. on Manual Control, Univ. of Southern California, Los Angeles, CA, June 1971.

[16]Wewerinke, P. H., "A Theoretical and Experimental Analysis of the Outside World Perception Process," Fourteenth Annual Conf. on Manual Control, Univ. of Southern California, Los Angeles, CA, April 1978.

[17]Wewerinke, P. H., "The Effect of Visual Information on the Manual Approach and Landing," Sixteenth Annual Conf. on Manual Control, Massachusetts Inst. of Technology, Cambridge, MA, May 1980, pp. 49–65.

[18]Grunwald, A. J., and Merhav, S. J., "Vehicular Control by Visual Field Cues—Analytical Model and Experimental Validation," *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 6, No. 12, 1976, pp. 835–845.

[19]Merhav, S. J., and Grunwald, A. J., "Display Augmentation in Manual Control of Remotely Piloted Vehicles," *Journal of Aircraft*, Vol. 15, No. 3, 1978, pp. 182–189.

[20]Grunwald, A. J., and Merhav, S. J., "Effectiveness of Basic Display Augmentation in Vehicular Control by Visual Field Cues," *IEEE Transactions on Systems, Man and Cybernetics*, Vol. 8, No. 9, 1978, pp. 679–690.

[21]Schmidt, D. K., and Silk, A. B., "Modeling Human Perception and Estimation of Kinematic Responses During Aircraft Landing," *Proceedings of the Guidance, Navigation, and Control Conference* (Minneapolis, MN), AIAA, Washington, DC, 1988, pp. 1117–1126.

JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 16, No. 2, March–April 1993

Designing Nonlinear Filters Based on Daum's Theory

Garfield C. Schmidt*
Boeing Defense and Space Group, Seattle, Washington 98124

The purpose of this paper is to present a method for designing nonlinear filters based on work by Frederick E. Daum. The evolution of a probability density function on an interval between measurements can be described by the Fokker-Plank equation that, under certain conditions, can be written as the product of a scalar function and an exponential function. The parameters defining the latter satisfy coupled ordinary differential equations and can be updated. However, it is very difficult to obtain the mean and covariance at this stage in the development of the theory. A major theoretical result communicated in this paper is the derivation of sufficient conditions, stated in terms of the nonlinear function defining the dynamic system, under which a probability density function exists satisfying Daum's conditions. This leads to algorithms for propagating the mean and covariance that generalize the Kalman-Bucy equations. A nonlinear filter for the exoatmospheric intercept of an intercontinental ballistic missile is given as an example.

I. Introduction

CONSIDER a dynamic system described by the stochastic differential equation

$$\dot{x}' = f(x', t) + \omega \tag{1}$$

Here x' is the system state vector and t time. The forcing function ω is a stochastic process. Measurements are taken at discrete time intervals and are functions of the states and another stochastic process v.

$$z = u(x') + v \tag{2}$$

The purpose of this paper is to present a method for designing nonlinear filters for such systems based on the work of Daum.[1–6] Daum's work is theoretically elegant, but it is difficult to generate computional algorithms based on the current state of the theory.

The portion of Daum's work that forms the basis for this paper will be summarized in Sec. II, along with a discussion of a new point of view providing for easier formulation of computational algorithms. The new conditions that, if satisfied, lead to computational algorithms are derived in Sec. III. The formulas for propagating the mean and covariance of the probability density function (pdf) for the estimates $\hat{x}'$ of x' are derived in Sec. IV. An example is given in Sec. V, and conclusions are summarized in Sec. VI.

II. New Perspective on Daum's Results

The specific formulation of Daum's results used here is presented in Ref. 2. Two of the assumptions that Daum makes are common to most developments. The first is

$$E\{\omega(t)\omega(\tau)^T\} = Q(t)\delta(t-\tau)$$
$$E\{\omega(t)\omega(\tau)^T\} = 0$$
$$E\{\omega(t)\omega(\tau)^T\} = R(t)\delta(t-\tau) \tag{3}$$

In Eqs. (3), E denotes expected value and δ the delta function. The matrices $Q(t)$ and $R(t)$ are called process and measurement noise matrices, respectively.

The second common assumption is that the measurements are linear in the state variables

$$z = H(t)x'(t) + v \tag{4}$$

where $H(t)$ is an $m \times n$ time-dependent matrix. Several observations may be made before proceeding to Daum's other conditions. It is well known[8] that the pdf $p(x,t)$ of the state variables defined by Eq. (1) satisfies the Fokker-Plank equation

$$\frac{\partial p}{\partial t} = -\left(\frac{\partial p}{\partial x'}\right) f - p\left[\mathrm{tr}\left(\frac{\partial f}{\partial x'}\right)\right] + \frac{1}{2}\mathrm{tr}\left(Q\frac{\partial^2 p}{\partial x'^2}\right) \tag{5}$$

in the interval (t_{k+1}, t_k) between measurement updates. Here, $\partial p/\partial x'$ is the gradient of p with respect to x' written as a row vector, $\partial f/\partial x'$ is the Jacobian of f with respect to x', tr denotes "trace," and $\partial^2 p/\partial x'^2$ is the Jacobian of the transpose of the gradient $\partial p/\partial x'$.

There are three elements common to recursive filters. The first is the assumption that the initial pdf $p(x,t_0)$ is defined in terms of a given set of parameters. The second is that the solution to Eq. (5) on (t_0,t_1) can be described by the same set of parameters. The third is that the update at t_1, based on the measurements at t_1, yields a pdf that again has the given form. One can then continue to propagate and update the pdf through as many measurements as necessary.

In Ref. 2, Daum assumes an unnormalized pdf of the form

$$\rho(x', t_0) = \Psi(x', t_0)^s e^{-\frac{1}{2}[(x'-m_0)^T P_0^{-1}(x'-m_0)]} \tag{6}$$

The vector m_0 is n dimensional, P_0 is an $n \times n$ positive-definite matrix, and s is a real number between 0 and 1. Conditions on $\Psi(x',t)$ will follow shortly. Only $s = \frac{1}{2}$ is considered here. An unnormalized pdf $\rho(x',t_0)$ can be made into a pdf by dividing it by the integral of the function over the whole state space. That is,

$$p(x', t_0) = \frac{\rho(x', t_0)}{\int \rho(x', t_0)\,\mathrm{d}x'} \tag{7}$$

The next observation is that since one assumes that $p(x',t_0)$ is known, the mean $\hat{x}'$ of the state vector at t_0 is known. Therefore, it is possible to express $f(x',t)$ as a series in the variable $x = (x' - \hat{x}')$. Assume that Eqs. (1), (2), (4), and (5) are written in terms of x, where $\hat{x}'$ is the mean of the state variables x' at the left-hand endpoint of the interval between two measurements.

Received June 17, 1991; presented as Paper 91-2700 at the AIAA Guidance, Navigation, and Control Conference, New Orleans, LA, Aug. 12–14, 1991; revision received Feb. 18, 1992; accepted for publication March 20, 1992.

*Principal Engineer, P.O. Box 3999, M/S 8R-03. Member AIAA.

Now let

$$r = \frac{\partial}{\partial x}\left[\ln \Psi(x,t)\right] \tag{8}$$

Then, Daum's other conditions can be stated as follows.

Daum's Conditions

Condition 1: $\Psi(x,t)$ also satisfies the Fokker-Plank equation

$$\frac{\partial \Psi(x,t)}{\partial t} = -\frac{\partial \Psi}{\partial x} f - \Psi\left[\operatorname{tr}\left(\frac{\partial f}{\partial x}\right)\right] + \frac{1}{2}\operatorname{tr}\left(Q\frac{\partial^2 \Psi}{\partial x^2}\right) \tag{9}$$

on (t_{k-1}, t_k).

Condition 2:

$$\operatorname{tr}\left(\frac{\partial f}{\partial x}\right) + (1/4) rQr^T = x^T A x + b^T x + c \tag{10}$$

for some symmetric matrix A, vector b, and scalar c.

Condition 3:

$$f - (1/2)Qr^T = Dx + E \tag{11}$$

for some matrix D and vector E.

In Ref. 2 Daum proves that if such a Ψ exists, then the unnormalized pdf at t_k conditioned on the set of measurements z_k is given by

$$p(x_k, t_k \mid z_k) = \Psi(x_k, t_k)^{1/2} e$$

$$\exp\left\{-\tfrac{1}{2}\left[(x_k - m_k)^T P_k^{-1}(x_k - m_k)\right]\right\} \tag{12}$$

The parameters m and P are propagated between measurements according to the following two differential equations:

$$\frac{dm}{dt} = -PAm + Dm - (1/2)Pb + E \tag{13}$$

$$\frac{dP}{dt} = -PAP + DP + PD^T + Q \tag{14}$$

The parameters before and after update are (−) and (+), respectively.

$$P_k(+)^{-1} m_k(+) = H^T R^{-1} z_k + P_k(-)^{-1} m_k(-) \tag{15}$$

$$P_k(+)^{-1} = H^T R^{-1} H + P_k(-)^{-1} \tag{16}$$

The parameters m and p are not the mean and covariance of the states, and it is difficult to build a recursive filter based on these results. At this point, it is profitable to ask the following question. Under what conditions will condition 1 have a solution, or an approximate solution, that is of the form dictated by conditions 2 and 3? Such conditions can be formulated in terms of the nonlinear function defining the dynamic system as shown in the next section.

III. Conditions for Daum's Solution to Exist

If we assume that f in Eq. (1) does not depend explicitly on t, f can be written as

$$f(x) = \hat{h} + Bx + U + \text{(higher-order terms)} \tag{17}$$

where

$$\hat{h} = f(\hat{x}') \tag{18}$$

$$B = \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}'} \tag{19}$$

$$U = \begin{bmatrix} x^T G_1 x \\ \vdots \\ x^T G_n x \end{bmatrix} \tag{20}$$

Each G_i is a symmetric matrix. Also,

$$\operatorname{tr}\left(\frac{\partial f}{\partial x}\right) = d + S^T x + x^T L x + \text{(higher-order terms)} \tag{21}$$

for some symmetric matrix L, vector S, and scalar d.

If conditions 2 and 3 are true up to and including second-order terms, then

$$\Psi(x,t) = e^{h(x,t)} \tag{22}$$

where $h(x,t)$ must have the form

$$h(x,t) = x^T V x + g^T x + \alpha(t) \tag{23}$$

for some symmetric matrix V, vector g, and scalar function $\alpha(t)$.

The partial differential equation for h is obtained by substituting Eq. (22) into Eq. (9).

$$\frac{\partial h}{\partial t} = -\frac{\partial h}{\partial x} f - \operatorname{tr}\left(\frac{\partial f}{\partial x}\right) + \tfrac{1}{2}\operatorname{tr}\left\{Q\left[\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^T\left(\frac{\partial h}{\partial x}\right)\right]\right\} \tag{24}$$

Using Eqs. (17) and (23), we get

$$\dot{\alpha}(t) = -(2x^T V + g^T)(\hat{h} + Bx + U) - d - S^T x - x^T L x$$

$$+ \operatorname{tr}(QV) + 2x^T VQVx + 2x^T VQg + \tfrac{1}{2}\operatorname{tr}(Qgg^T)$$

$$+ \text{(higher-order terms)} \tag{25}$$

Define $J(x,t)$ as follows:

$$J(x,t) = x^T(2VQV - 2VB - L - g_1 G_1 - \cdots - g_n G_n)x$$

$$+ (2g^T QV - g^T B - 2\hat{h}^T V - S^T)x + \tfrac{1}{2}\operatorname{tr}(Qgg^T + 2QV)$$

$$- g^T \hat{h} - d - \dot{\alpha}(t) \tag{26}$$

If V and g can be chosen so that $J(x,t)$ is identically zero in (t_{k-1}, t_k) then h as defined by Eq. (23) is an approximate solution of Eq. (24) up to the second order. Conditions for the existence of a second-order approximation of Eq. (9) satisfying conditions 2 and 3 are given by setting each term of Eq. (26) equal to zero. These are the following.

Conditions A

$$2(VQV - VB) = L + (g_1 G_1 + \cdots + g_n G_n) \tag{27}$$

$$g^T(2QV - B) = S^T + 2\hat{h}^T V \tag{28}$$

$$\dot{\alpha}(t) = \tfrac{1}{2}\operatorname{tr}(Qgg^T + 2QV) - g^T \hat{h} - d \tag{29}$$

If f is linear or a linear approximation is used, then Eqs. (27) and (28) are trivially satisfied with $V = 0$ and $g = 0$. Linear and extended Kalman filters fall under this category.

A weaker set of conditions is obtained if one only requires that the value of J at (x_{k-1}, t_{k-1}) be zero, that $\partial J/\partial x = 0$ in a neighborhood of x_{k-1}, and that $\partial J/\partial t = 0$ at (x_{k-1}, t_{k-1}). Then J will be a zero in a neighborhood of (x_{k-1}, t_{k-1}). Thus, h defined by Eq. (23) will be a first-order solution of Eq. (24). From Eq. (26), we have

$$\frac{\partial J}{\partial x} = -2x^T(VB + B^T V) + 4x^T VQV$$

$$- 2x^T(L + g_1 G_1 + \cdots + g_n G_n)$$

$$+ 2g^T QV - g^T B - 2\hat{h}^T V - S^T \tag{30}$$

The conditions for the existence of a first-order solution to Eq. (9) satisfying conditions 2 and 3 may now be written as follows.

Conditions B

$$2VQV - (VB + B^T V) = \left[L + (g_1 G_1 + \cdots + g_n G_n)\right] \tag{31}$$

$$g^T(2QV - B) = S^T + 2\hat{h}^T V \tag{32}$$

$$\dot{\alpha}(t) = \beta(t) \tag{33}$$

Note that $J(x,t) = 0$ over (t_{k-1}, t_k) does not imply either condition A or B.

IV. Propagation and Update

The relationship between Daum's parameters P and m and the covariance M and mean $\hat{x}$ can now be established. From Eqs. (6), (22), and (23), it follows that

$$p(x,t) = C_1 e^{-\frac{1}{2}\left[(x-m)^T P^{-1}(x-m) - (x^T V x + g^T x)\right]} \tag{34}$$

where C_1 depends on time alone. Equation (34) may be rewritten as

$$p(x,t) = C_2 e^{-\frac{1}{2}(x-\hat{x})^T M^{-1}(x-\hat{x})} \tag{35}$$

where

$$M^{-1} = P^{-1} - V \tag{36}$$

$$\hat{x} = MP^{-1} m + \tfrac{1}{2} Mg \tag{37}$$

Distribution (35) is Gaussian. The covariance M and mean $\hat{x}$ are propagated using the formulas derived next.

Daum's equation (14) can be written

$$\frac{d}{dt}(P^{-1}) = A - P^{-1}D - D^T P^{-1} - P^{-1} Q P^{-1} \tag{38}$$

From Eqs. (8), (10), (11), (17), and (23), we find that

$$A = L + VQV \tag{39}$$

$$b^T = S^T + g^T QV \tag{40}$$

$$c = d + (1/4) g^T Q g \tag{41}$$

$$D = B - QV \tag{42}$$

$$E = \hat{h} - (1/2) Qg \tag{43}$$

It can now be shown that

$$\dot{M} = BM + MB^T - M(2L + g_1 G_1 + \cdots + g_n G_n)M + Q \tag{44}$$

This equation for propagating the covariance differs from that of an EKF by the factor $M(2L + g_1 G_1 + \cdots + g_n G_n)M$.

To obtain the formula for propagating the mean, first multiply both sides of Eq. (37) by PM^{-1}.

$$PM^{-1}\hat{x} = m + \tfrac{1}{2} Pg \tag{45}$$

Differentiate Eq. (45) and simplify to show that

$$\dot{\hat{x}} = \hat{h} + B\hat{x} - M(2L + g_1 G_1 + \cdots + g_n G_n)\hat{x} - MS \tag{46}$$

This formula for propagating the mean differs from that of an EKF by the factors $M(2L + g_1 G_1 + \cdots + g_n G_n)\hat{x}$ and MS.

The equations for updating M and $\hat{x}$ are obtained by substituting Eqs. (36) and (37) into Eqs. (15) and (16). The matrix V and vector g cancel out of these equations, leaving the same form with P and m replaced by M and $\hat{x}$. (See Ref. 7).

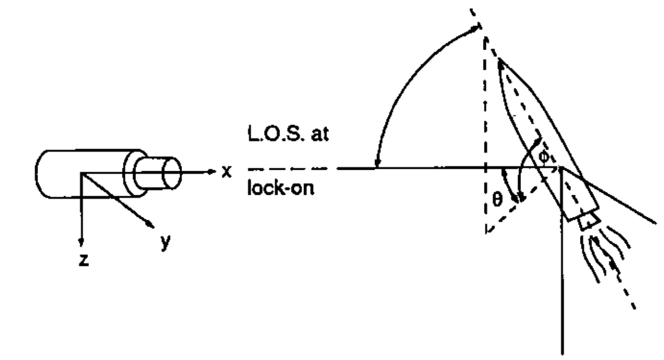

Fig. 1 Engagement geometry.

V. State Equations for Guidance Filters

A nonlinear filter is developed to illustrate the result of the preceding section. Consider the engagement of a space-based interceptor against an intercontinental ballistic missile (ICBM). The interceptor has an angle-only seeker with a given acquisition range. A coordinate system, called the terminal homing coordinate system (THCS), is defined by taking the x axis to be the line of sight from the interceptor to ICBM at acquisition, and the y and z axes to be the y and z body axes of the interceptor at acquisition. The THCS moves with the interceptor, but does not rotate in inertial space. The interceptor is equipped with rate gyros so that attitude of the interceptor body axis relative to the THCS can be estimated. The two angle measurements taken relative to the seeker boresight can therefore be transformed into two angle measurements, azimuth and elevation, in the THCS. The azimuth angle lies in the x,y plane and remains small throughout terminal homing, so no loss of performance results if the elevation angle is thought of as lying in the x,z plane. Both measured angles remain small because of the limited divert capability of the interceptor. Thus, the measurement equation is essentially linear.

The x body axis of the target ICBM is along its length (axis of symmetry) and is oriented relative to the THCS by yawing through θ deg from the negative x axis and then pitching through ϕ deg as indicated in Fig. 1. (The pitch angle in Fig. 1 is negative, the yaw angle positive.)

Two filters identical in structure are implemented: one for the x,y plane and one for the y,z plane. The x,y plane filter equations will be derived here. Let T denote thrust, m mass, and I_{sp} the specific impulse of the ICBM. Also, let c_y denote the direction cosine of the target acceleration with respect to the y axis. The states for the x,y plane are defined as follows.

$$y_1 = \text{azimuth angle}$$

$$y_2 = \text{azimuth angle rate}$$

$$y_3 = c_y(T/m)$$

$$y_4 = T/(gI_{sp}m) \tag{47}$$

Using the fact that, for a rocket,

$$\dot{m} = -T/(gI_{sp}) \tag{48}$$

one sees that

$$\dot{y}_3 = y_3 y_4 + \dot{c}_y(T/m) \tag{49}$$

$$\dot{y}_4 = y_4^2 \tag{50}$$

The nonlinearities in the filter equation are due to the way in which target acceleration is described in Eqs. (49) and (50) and not in the choice of a coordinate system.

If the range and range rate are R and $\dot{R}$, then the following states, which are called range axis states, appear in the filter equations. The direction cosine of target acceleration is denoted c_x.

$$x_1 = \dot{R}/R$$

$$x_2 = 1/R$$

$$x_3 = c_x(T/m)$$

$$x_4 = T/(gI_{sp}m) \tag{51}$$

The seeker does not provide a range measurement, and these range axis states are not observable unless the interceptor is caused to fly particular types of trajectories.[9] These trajectories will not be considered here. An initial estimate of the range axis state is made that contains errors (quantified later), and the range axis states are propagated without update. These errors must be small; otherwise the filter diverges.

The x and y coordinates of the target in the THCS are

$$y = R \sin y_1 \tag{52}$$

$$x = R \cos y_1 \tag{53}$$

Differentiating twice, using small angle approximations, and simplifying yield the equation for $\dot{y}_2$. Because the degree-of-target maneuver reflected in $\dot{c}_y$ is not known, the term $\dot{c}_y(T/m)$ is represented by a stochastic process ω_1. A stochastic process ω_2 is also added to the equation for y_4 because of the uncertainty in I_{sp} and lack of knowledge of (T/m) at the start of homing. The filter-state equations are thus

$$\dot{y}_1 = y_2$$

$$\dot{y}_2 = -x_2x_3y_1 - 2x_1y_2 + x_2(y_3 - A_{Iy})$$

$$\dot{y}_3 = y_3y_4 + \omega_1$$

$$\dot{y}_4 = y_4^2 + \omega_2 \tag{54}$$

where A_{Iy} is the interceptor's own acceleration in the xy plane.

Expand the right side of the state equations (54) about the estimated mean (now denoted $\hat{y}$ rather than $\hat{x}$). It follows that

$$\hat{n} = \begin{bmatrix} \hat{y}_2 \\ -x_2x_3\hat{y}_1 - 2x_1\hat{y}_2 + x_2\hat{y}_3 - x_2A_{IY} \\ \hat{y}_3\hat{y}_4 \\ \hat{y}_4^2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -x_2x_3 & -2x_1 & x_2 & 0 \\ 0 & 0 & \hat{y}_4 & \hat{y}_3 \\ 0 & 0 & 0 & 2\hat{y}_4 \end{bmatrix} \tag{55}$$

$$L = [0]$$

$$S^T = [0, 0, 0, 3]$$

$$d = -2x_1 + 3\hat{y}_4$$

Furthermore, G_1 and G_2 are zero, G_3 has 0.5 in the (3, 4) and (4, 3) positions, G_4 has 1 in the (4, 4) position, and Q has $q_1(t)$ and $q_2(t)$ in the (3, 3) and (4, 4) positions. Otherwise, all elements of G_3, G_4, and Q are zero.

Conditions A are satisfied by a vector g that is zero except for the fourth element that we denote by g again, and a matrix V that is zero except for the (4, 4) element, which we label ν. The matrix products in Eq. (27) all have zero entries except for the (4, 4) element. The scalar equation for this position is

$$g = 2(q_2\nu^2 - 2\hat{y}_4\nu) \tag{56}$$

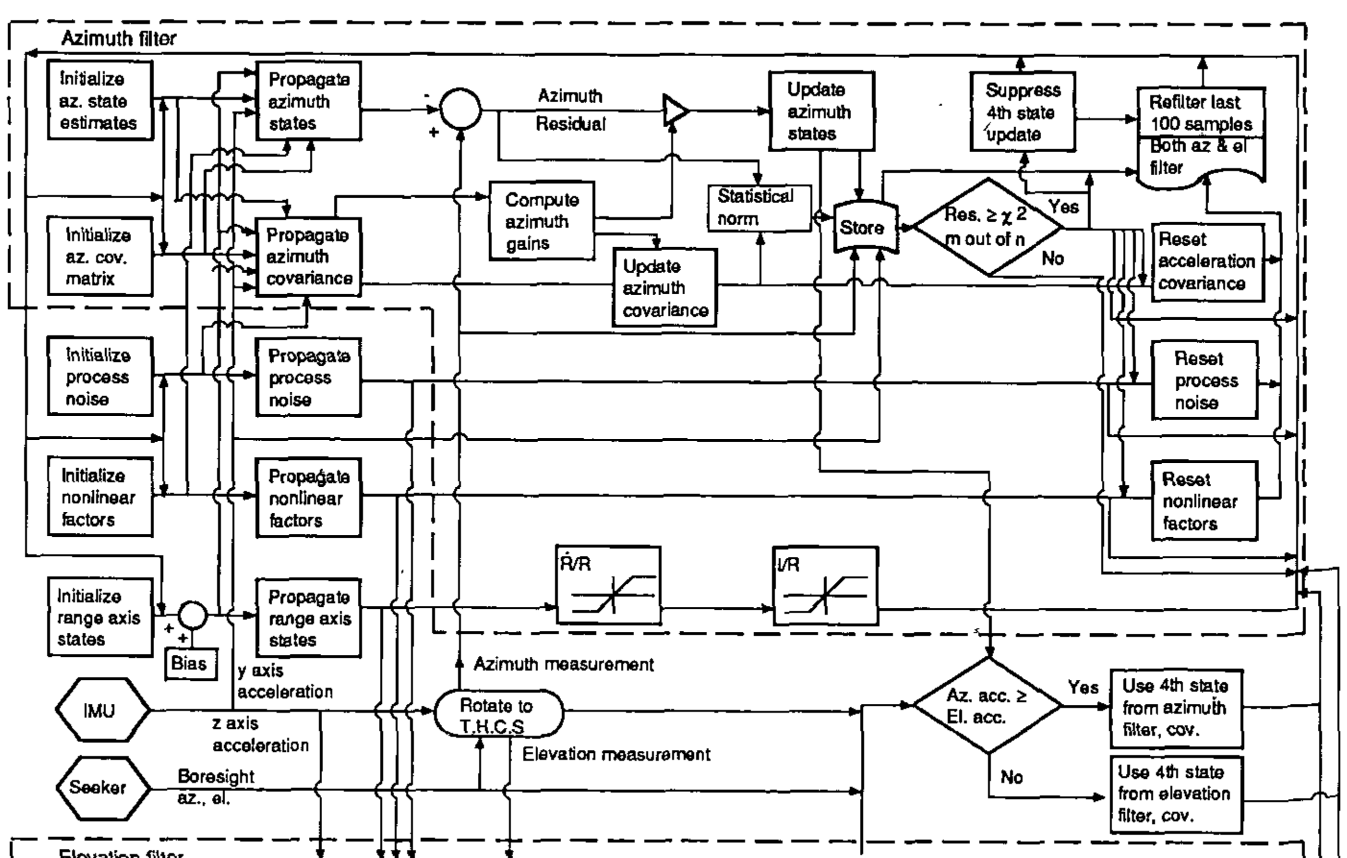

Fig. 2 New nonlinear filter block diagram.

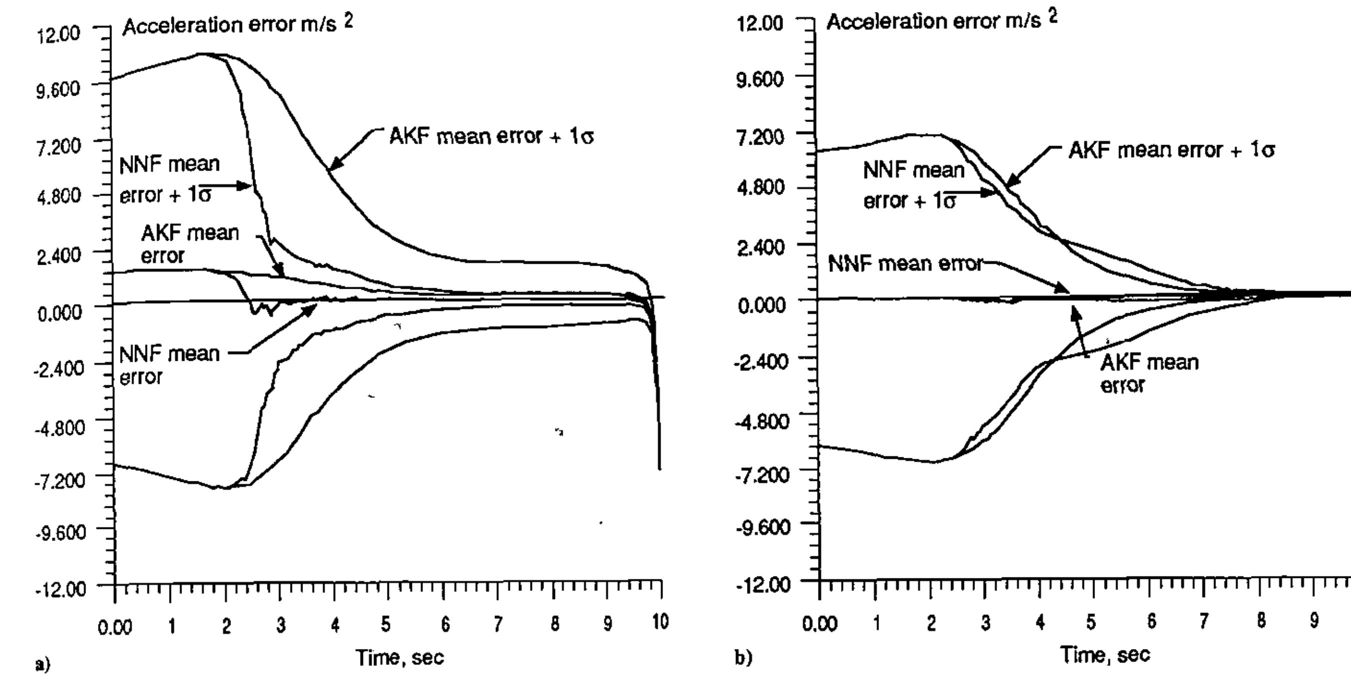

Fig. 3 Mean and 1σ acceleration estimate errors: a) x,y plane (azimuth) and b) x,z plane (elevation).

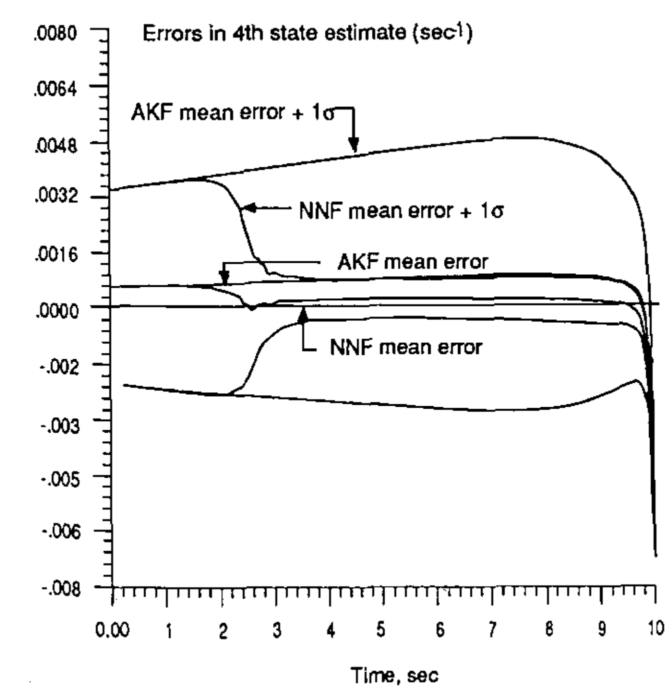

Fig. 4 Mean and 1σ fourth-state estimate errors.

The vector equation (28) has zeros in all components except the fourth. The scalar equation for the fourth element is

$$g = (3 + 2\hat{y}_4^2\nu)/2(q_2\nu - \hat{y}_4) \tag{57}$$

It is customary in filter design to choose process noise as one of the parameters to vary to improve filter performance. The method used here is admittedly heuristic. The best filter performance has been obtained by using ν as a parameter and computing q_2 and g via the following formula that is derived from Eqs. (56) and (57):

$$g_1 = -\hat{y}_4\nu_1 - \sqrt{3\nu_1(1 + \nu_1\hat{y}_4^2)} \tag{58}$$

$$g_2 = -\hat{y}_4\nu_2 - \sqrt{3\nu_2(1 + \nu_2\hat{y}_4^2)} \tag{59}$$

$$q_2 = \frac{2\hat{y}_4}{\nu_2} + \frac{g_2}{2\nu_2^2} \tag{60}$$

The parameters ν_1 and ν_2 in formulas (58–60) were chosen to be exponentially decreasing functions with a time constant of one. The parameter g_1 is used in all covariance propagating equations except the tenth [or (4, 4) in matrix terminology], which uses g_2.

A block diagram illustrating the implementation of both the new nonlinear filter (NNF) and an adaptive Kalman filter (AKF) is given as Fig. 2. The elevation filter is not included in detail because it is similar to the azimuth filter. However, the filters are somewhat coupled as indicated in the diagram.

Both the NNF and AKF use the same maneuver detection criteria. The residuals for both filters are normalized by dividing by the appropriate angle covariance and stored along with the data necessary to refilter a portion of the measurements once a maneuver is detected. When these residuals in either filter exceed a given value in m out of n times, a maneuver detection switch is set. Before refiltering the stored data, the q_1 process noise and acceleration covariance for each filter are reinitialized. These terms in both filters are reinitialized to the same value. Of course, the NNF has additional terms in the covariance matrix propagation, and the best way to make use of these terms appears to be reinitialization of the (3, 4) cross-covariance term. After a maneuver detection, the update of the fourth state is suppressed because a change in $T/(gI_{sp}m)$ due to a change in acceleration direction cannot be distinguished from that due to the rocket equation, which is really what it is supposed to reflect.

Since maneuver detection is being done, the value of process noise is chosen to be a decreasing exponential function. The initial value and time constant are chosen for best AKF performance after maneuver detection and a steady-state value is chosen for best AKF steady-state performance. The same process noise is used in both filters and the same initial covariance matrix is used for both filters. Therefore, the improvement the NNF shows over the AKF comes only from designing with the additional terms suggested by Eqs. (44) and (46).

The interceptor and target are closing at 10,000 m/s and the engagement lasts 10 s. The y and z axis 1σ errors in initial position, velocity, and acceleration were 750 m, 75 m/s, and 7.5 m/s^2, respectively. Perfect range axis information was used to generate the data of Figs. 3 and 4 to illustrate the difference between filters. Range axis errors equal to the y and z axes errors were used in Fig. 5. The target acceleration at impact was 120 m/s^2 and the zero effort miss 2386 m in each axis. The seeker tracked a point in the ICBM target plume, and an aimpoint bias was used to move the intended impact from this point in the plume to the target center. The seeker accuracy was 100 μrad at 100 km coming down to a floor of 50 μrad. For miss distance computation and probability of hit calculations, the ICBM body was 3 m in diameter and 13 m long. The interceptor had a 100-m/s^2 acceleration limit.

The acceleration occurs along the y axis to illustrate filter performance over the widest range of accelerations, i.e., large accelerations in the azimuth filter and no acceleration in the elevation filter. The most significant difference in performance occurs in the third- and fourth-state estimates, as illustrated in Figs. 3 and 4. This is to be expected, since this is where the nonlinearities occur. The NNF performs better by converging faster and having smaller 1σ steady-state errors. When the target acceleration is partitioned between the y and z axes, the fourth-state convergence is not as good, but it is still better than the AKF and convergence in the acceleration states remained better than for the AKF.

The following guidance law was used:

$$U_{cy} = (1 + at_g^2)\Delta\left[V_c\hat{y}_2 - bx_1x_2y_b + (0.5)\hat{y}_3\right] \tag{61}$$

where U_{cy} is acceleration in the xy plane. A similar formula holds for the xz plane. Here Δ, a, and b are constants, V_c is the closing velocity, and the aimpoint bias term is $bx_1x_2y_b$. Time to go is t_g. There are a number of important issues concerning the implementation of the guidance loop that cannot be covered here. The NNF shows improvement (in terms of probability of hit) over the AKF only when the target maneuvers and an aggressive guidance law are necessary to take advantage of this.

Probability of hit vs maneuver angle for discrete times starting at 8 s (2 s before impact) is presented in Fig. 5. Here a

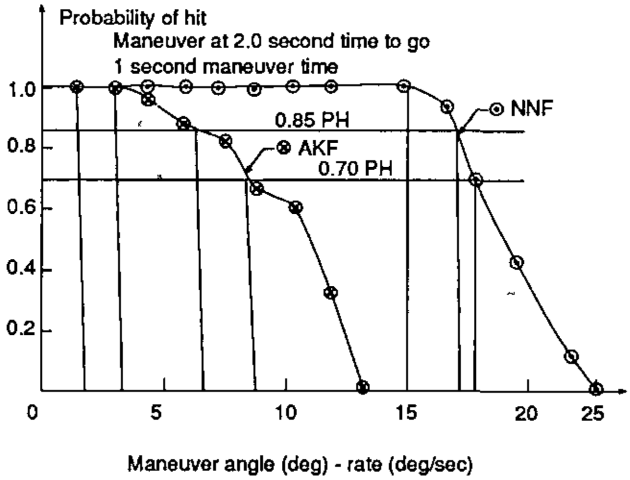

Fig. 5 Performance against a discrete maneuver.

discrete turn is one where the target starts with a rotation rate of zero, the nozzle is gimballed for 1 s during which the target rotates through a given angle, and then the rotation rate is zeroed out via a control law. The target starts out accelerating along the y axis and then yaws, so all of the acceleration remains perpendicular to the line of sight. This case best illustrates the difference between NNF and AKF performance.

VI. Conclusion

This paper presents a new mathematical foundation for designing nonlinear filters. Conditions stated in terms of the state equations defining the dynamic system are derived, under which the Kalman-Bucy equations for propagating the state estimates and covariance may be generalized. The interpretation of the additional terms that appear in the state and covariance propagation equations and the way in which one can use these terms to improve filter performance are far from clear. However, we have presented an example that demonstrates significant performance improvement can be obtained using these nonlinear terms and therefore justifies calling the attention of the filtering and tracking community to these results. It is also significant, at least theoretically, that in some cases the pdf remains Gaussian even if the filter equations are nonlinear.

Acknowledgment

The results reported in this paper were obtained under Air Force Contract F08635-89-C-0096, "Nonlinear Estimation of Exoatmospheric Trajectories."

References

[1]Daum, F. E., "Application of New Nonlinear Filtering Theory," *Proceedings of the American Control Conference* (Minneapolis, MN), IEEE Computer Society, 1987, pp. 2153–2154.

[2]Daum, F. E., "Exact Finite Dimensional Nonlinear Filters," *IEEE Transactions on Automatic Control,* Vol. AC-31, No. 7, July 1986, pp. 616–621.

[3]Daum, F. E., "New Nonlinear Filters and Exact Solutions of the Fokker-Plank Equation," 5th American Control Conference (Seattle, WA), American Automatic Control Council and IEEE Computer Society, June 1986.

[4]Daum, F. E., "Exact Finite Dimensional Nonlinear Filters," *Proceedings of the 24th IEEE Conference on Decision Control* (Fort Lauderdale, FL), IEEE Computer Society, Dec. 1985, pp. 1938–1945.

[5]Daum, F. E., "A New Nonlinear Filtering Formula for Discrete Time Measurements," *Proceedings of the 24th IEEE Conference on Decision Control* (Fort Lauderdale, FL), IEEE Computer Society, Dec. 1985, pp. 1957–1958.

[6]Daum, F. E., "Exact Solution to the Zakai Equation for Certain Diffusions," *Proceedings of the 24th IEEE Conference on Decision Control* (Fort Lauderdale, FL), IEEE Computer Society, Dec. 1985, pp. 1964–1965.

[7]Gelb, A., *Applied Optimal Estimation,* MIT Press, Cambridge, MA, 1974.

[8]Jazwinski, A. H., *Stochastic Processes and Filtering Theory,* Academic, New York, 1970.

[9]Speyer, J. L., Hull, D. G., and Yseng, C. Y., "Estimation Enhancement by Trajectory Modulation for Homing Missiles," *Journal of Guidance, Control, and Dynamics,* Vol. 7, No. 2, 1984, pp. 167–174.